Anisotropic function spaces, anisotropic fractals, and spectral theory for related fractal semi-elliptic operators

Dissertation

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Contents

Zusammenfassung

Die Theorie der anisotropen Funktionenräume entwickelte sich parallel zur Theorie von isotropen Funktionenräumen. Wir verweisen insbesondere auf Arbeiten von S.M. Nikol'skiĭ, O.V. Besov.

Die anisotropen Funktionenräume erscheinen dann, wenn man Differentialoperatoren untersucht, deren maximale Ableitungsordnungen verschieden von Richtung zu Richtung sind, z.B. der Operator der Wärmeleitungsgleichung. Falls $1 < p < \infty$ und $(s_1, \ldots, s_n)$ ein $n-$Tupel von natürlichen Zahlen sind, dann ist

$$\begin{aligned} W_p^{(s_1,\ldots,s_n)}(\mathbb{R}^n) &= W_p^{s,a}(\mathbb{R}^n) \\ &= \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|L_p(\mathbb{R}^n)\| + \sum_{k=1}^{n} \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} |L_p(\mathbb{R}^n)\right\| < \infty \right\} \end{aligned}$$

der klassische anisotrope Sobolev-Raum auf $\mathbb{R}^n$. In Vergleich zum isotropen Sobolev-Raum $(s_1 = \cdots = s_n)$ sind die Regularitätseigenschaften einer Funktion aus $W_p^{s,a}(\mathbb{R}^n)$ von der in $\mathbb{R}^n$ ausgewählten Richtung abhängig. Die Zahl s, die durch

$$\frac{1}{s} = \frac{1}{n}\left(\frac{1}{s_1} + \cdots + \frac{1}{s_n}\right),$$

definiert ist, wird gewöhnlich als 'mittlere Glattheit' bezeichnet; $a = (a_1, \ldots, a_n)$ bezeichnet die 'Anisotropie', wobei

$$a_1 = \frac{s}{s_1}, \ \ldots, \ a_n = \frac{s}{s_n}.$$

In der vorliegenden Arbeit werden Zusammenhänge zwischen fraktaler Geometrie und der Fourieranalysis, der Theorie der Funktionenräume sowie der Spektraltheorie einiger Differentialoperatoren untersucht.

Die Arbeit hat fünf Teile. Im ersten Kapitel stellen wir Grundlagen für anisotrope Besov-Räume zusammen. Wir verwenden die Fourier-analytische Definition von $B_{pq}^{s,a}(\mathbb{R}^n)$: eine Funktion $f \in \mathcal{S}'(\mathbb{R}^n)$ wird in eine Summe von ganzen analytischen Funktionen $(\varphi_j \widehat{f})^\vee$ zerlegt, die dann bezüglich ℓ_q und $L_p(\mathbb{R}^n)$, gemessen werden.

Das zweite Kapitel widmet sich einigen wichtigen Eigenschaften der anisotropen Besov-Räume.

Das dritte Kapitel beschäftigt sich mit Zerlegungen (Atome, Wavelets) in anisotropen Funktionenräumen. Unser Hauptziel in diesem Kapitel ist, das anisotrope Gegenstück zu einem Resultat von H.Triebel(2003) zu beweisen. Der Hauptpunkt unserer Untersuchung ist die Kombination der Waveletphilosophie mit der Idee der Taylor-Entwicklung, d.h.

$$f(x) \to f(2^{ja}x - m) \quad \text{mit} \quad f(x) \to x^{\beta} f(x),$$

für $j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n$ und $\beta \in \mathbb{N}_0^n$. Dann erhält man einfache explizite Wavelet-Darstellungen, die gleichzeitig globales und lokales Verhalten charakterisieren. In Kapitel 4 geben wir die Definition der anisotropen $d-$Mengen; das sind z.B. anisotrope Cantor-Mengen. Wir studieren die Existenz und die Eigenschaften des Spur Operators tr_Γ, zwischen den Funktionenräumen, basierend auf Wavelet-Darstellungen aus Kapitel 3. Damit erhalten wir asymptotisch scharfe Abschätzungen der Approximationzahlen für den kompakten Spuroperator tr_Γ,

$$a_k(tr_\Gamma : B^{s,a}_{pp}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)) \ \sim\ k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}, \quad \frac{n}{p} \geq s > \frac{n-d}{p}.$$

Im letzten Kapitel betrachten wir den semi-elliptischen Differentialoperator

$$Au(x) = (-1)^{s_1}\frac{\partial^{2s_1}u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n}\frac{\partial^{2s_n}u(x)}{\partial x_n^{2s_n}} + u(x),$$

wobei $s_1, \ldots, s_n \in \mathbb{N}$ und $\frac{1}{s} = \frac{1}{n}(\frac{1}{s_1} + \cdots + \frac{1}{s_n})$. Der Operator $A^{-1} \circ tr^\Gamma$ ist in $W^{s,a}_2(\mathbb{R}^n)$ eine kompakte, nicht-negative selbstadjungierte Abbildung. Für seine Eigenwerte, gezählt entsprechend ihrer Vielfachheit und monoton geordnet, enthält man mit Hilfe der Ergebnisse aus Kapitel 4,

$$c_1 k^{-\frac{1}{d}(d+2s-n)} \leq \lambda_k(A^{-1} \circ tr^\Gamma) \leq c_2 k^{-\frac{1}{d}(d+2s-n)}, \quad k \in \mathbb{N}.$$

Diese Resultate werden abschließend mit ähnlichen, bereits bekannten (Farkas, 2001) verglichen.

Introduction

The theory of the anisotropic spaces has been developed from the very beginning parallel to the theory of isotropic function spaces. We refer in particular to the Russian school and works of S.M. Nikol'skiĭ, O.V. Besov, V.P.Il'in [3, 35].
Let $1 < p < \infty$ and $(s_1, ..., s_n)$ be an n- tuple of natural numbers. Then

$$W_p^{s,a}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|L_p(\mathbb{R}^n)\| + \sum_{k=1}^{n} \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} |L_p(\mathbb{R}^n) \right\| < \infty \right\}$$

is the classical anisotropic Sobolev space on $\mathbb{R}^n$. It is obvious that unlike in case of the usual (isotropic) Sobolev space ($s_1 = ... = s_n$) the smoothness properties of an element from $W_p^{s,a}(\mathbb{R}^n)$ depend on the chosen direction in $\mathbb{R}^n$. The number s, defined by

$$\frac{1}{s} = \frac{1}{n} \left(\frac{1}{s_1} + \cdots + \frac{1}{s_n} \right),$$

is usually called the *mean smoothness*, and $a = (a_1, ..., a_n)$,

$$a_1 = \frac{s}{s_1}, \ \ldots, \ a_n = \frac{s}{s_n}$$

characterises the *anisotropy*. Similar to the isotropic situation the more general anisotropic Bessel potential spaces (fractional Sobolev spaces) $H_p^{s,a}(\mathbb{R}^n)$, where $1 < p < \infty$, $s \in \mathbb{R}$ and $a = (a_1, ..., a_n)$ is a given anisotropy, fit in the scales of anisotropic Besov spaces $B_{pq}^{s,a}(\mathbb{R}^n)$, and anisotropic Triebel-Lizorkin spaces $F_{pq}^{s,a}(\mathbb{R}^n)$, respectively. It is well known that this theory has a more or less complete counterpart to the basic facts (definitions, description via differences and derivatives, elementary properties, embeddings for different metrics, interpolation) of isotropic spaces $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$.

The purpose of this work is to highlight some aspects concerning the close connection between fractal geometry, the theory of function spaces, Fourier analysis, and spectral theory of differential operators.
The thesis has five parts. In the **first chapter** we collect fundamentals about anisotropic Besov spaces. We shall use the Fourier-analytical definition of

$B^{s,a}_{pq}(\mathbb{R}^n)$, where any function $f \in \mathcal{S}'(\mathbb{R}^n)$ is decomposed in a sum of entire analytic functions $(\varphi_j \widehat{f})^\vee$ and this decomposition, measured in ℓ_q and $L_p(\mathbb{R}^n)$, respectively, is used to introduce the spaces. This concept goes back to [44] and [43], see [37, Chapter 4].

The **second chapter** is devoted to some important properties of the spaces $B^{s,a}_{pq}$. In order to show the main results in chapter 3 and 4, we give in this chapter some equivalent quasi-norms in $B^{s,a}_{pq}$, the anisotropic counterpart to homogeneity estimate and at last the localization property in $B^{s,a}_{pp}$.

The **third chapter** deals with decompositions in anisotropic function spaces. Several authors were concerned in the last years with the problem of obtaining useful decompositions of anisotropic function spaces, too. A construction of unconditional bases in $B^{s,a}_{pq}(\mathbb{R}^n)$ and $F^{s,a}_{pq}(\mathbb{R}^n)$ spaces using Meyer wavelets was obtained in [1], [2]; see, more recently, [24], [23], [25]; a different approach, involving the φ-transform of Frazier and Jawerth (see [20], [21]) was followed in [9], [10], see also [38]. The most recent contributions we know are made in [6], [28], [29]; see also [4], [5].

Our main aim in this chapter is to prove an anisotropic counterpart of some recent results on wavelet frames parallel to the isotropic case in [52]. The main point of our approach is the combination of the wavelet philosophy with the Taylor-expansion philosophy, that is,

$$f(x) \to f(2^{ja}x - m) \quad \text{with} \quad f(x) \to x^\beta f(x), \quad \text{respectively,}$$

for $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $\beta \in \mathbb{N}_0^n$. Then one gets comparatively simple explicit wavelet frames which reflect simultaneously global and local behaviour.

As a first goal in **chapter** 4 we define an anisotropic $d-$set, we study the existence and properties of the trace operator acting between function spaces based on wavelet frames. The main aim in this chapter is to present a new method to estimate approximation numbers of compact trace operator as an application of our wavelet decomposition. Namely, it turns out that

$$a_k(tr_\Gamma : B^{s,a}_{pp}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)) \;\sim\; k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}, \quad \frac{n}{p} \geq s > \frac{n-d}{p},$$

as in the isotropic case, see [54].

In the **last chapter** we consider the semi-elliptic differential operator

$$Au(x) = (-1)^{s_1}\frac{\partial^{2s_1}u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n}\frac{\partial^{2s_n}u(x)}{\partial x_n^{2s_n}} + u(x)$$

and $tr^\Gamma = id_\Gamma \circ tr_\Gamma$. The main object of this chapter is to study spectral properties of the operator $A^{-1} \circ tr^\Gamma$ acting in the anisotropic Sobolev spaces

$W_2^{s,a}(\mathbb{R}^n)$. We will show that $A^{-1} \circ tr^\Gamma$ is compact, non-negative, and self-adjoint in $W_2^{s,a}(\mathbb{R}^n)$ and that there exist constants $c_1, c_2 > 0$ such that its positive eigenvalues, repeated according to multiplicity and ordered by their magnitude, can be estimated by

$$c_1 k^{-\frac{1}{d}(d+2s-n)} \leq \lambda_k(A^{-1} \circ tr^\Gamma) \leq c_2 k^{-\frac{1}{d}(d+2s-n)}, \quad k \in \mathbb{N}.$$

Finally, we compare results obtained by this method with the method obtained by Farkas [18].

1 Anisotropic Besov spaces

1.1 General notation

As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ stands for the complex numbers, and $\mathbb{Z}^n$ means the lattice of all points in $\mathbb{R}^n$ with integer-valued components. We use the equivalence "$\sim$" in $\varphi(x) \sim \psi(x)$ always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\,\varphi(x) \leq \psi(x) \leq c_2\,\varphi(x)$$

for all admitted values of x, where φ, ψ are non-negative functions. If $a \in \mathbb{R}$ then $a_+ := \max(a, 0)$. Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ be a multi-index, then

$$|\alpha| = \alpha_1 + ... + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \qquad \alpha \in \mathbb{N}_0^n, \tag{1.1}$$

the derivatives D^α have the usual meaning, x^α means $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$, and $\alpha\gamma = \alpha_1\gamma_1 + \cdots + \alpha_n\gamma_n$, $\gamma \in \mathbb{R}^n$, stands for the scalar product in $\mathbb{R}^n$.
Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula. We shall mainly deal with function spaces on $\mathbb{R}^n$; so for convenience we shall usually omit the "$\mathbb{R}^n$" from their notation, if there is no danger of confusion.

1.2 Anisotropic distance function

Let $a = (a_1, \ldots, a_n)$ be a fixed $n-$tuple of positive numbers with $a_1 + \cdots + a_n = n$, then we call a an **anisotropy**. We shall denote $a_{\min} = \min\{a_i : 1 \leq i \leq n\}$ and $a_{\max} = \max\{a_i : 1 \leq i \leq n\}$. If $a = (1, \ldots, 1)$ we speak about the "isotropic case".
The action of $t \in [0, \infty)$ on $x \in \mathbb{R}^n$ is defined by the formula

$$t^a x = (t^{a_1} x_1, \ldots, t^{a_n} x_n). \tag{1.2}$$

For $t > 0$ and $s \in \mathbb{R}$ we put $t^{sa}x = (t^s)^a x$. In particular we write $t^{-a}x = (t^{-1})^a x$ and $2^{-ja}x = (2^{-j})^a x$.

Definition 1.2.1 *An* ***anisotropic distance function*** *is a continuous function $u : \mathbb{R}^n \to \mathbb{R}$ with the properties $u(x) > 0$ if $x \neq 0$ and $u(t^a x) = tu(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$.*

Remark 1.2.2 *It is easy to see that $u_\lambda : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$u_\lambda(x) = \left(\sum_{i=1}^{n} |x_i|^{\frac{\lambda}{a_i}} \right)^{1/\lambda} \tag{1.3}$$

is an anisotropic distance function for every $0 < \lambda < \infty$, u_2 is usually called the anisotropic distance of x to the origin, see [37, Sect. 4.2.1]. It is well known, see [10, Sect. 1.2.3] and [57, Sect. 1.4], that any two anisotropic distance functions u and u' are equivalent (in the sense that there exist constants $c, c' > 0$ such that $cu(x) \leq u'(x) \leq c'u(x)$ for all $x \in \mathbb{R}^n$) and that if u is an anisotropic distance function there exists a constant $c > 0$ such that $u(x+y) \leq c(u(x)+u(y))$ for all $x, y \in \mathbb{R}^n$. We are interested to use smooth anisotropic distance functions. Note that for appropriate values of λ one can obtain arbitrary (finite) smoothness of the function u_λ from (1.3)*, cf. [10, Sect. 1.2.4]. A standard method concerning the construction of anisotropic distance functions in $C^\infty(\mathbb{R}^n \backslash \{0\})$ was given in [42].*

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $x \neq 0$, let $|x|_a$ be the unique positive number t such that

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} = 1 \tag{1.4}$$

and let $|0|_a = 0$; then $|\cdot|_a$ is an anisotropic distance function in $C^\infty(\mathbb{R}^n \backslash \{0\})$, see [57, Sect. 1.4/3,8]. Plainly, $|x|_a$ is in the isotropic case the Euclidean distance of x to the origin.

1.3 Anisotropic function spaces

Before introducing the function spaces under consideration we need to recall some notation. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with

$0 < p \le \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\|f \mid L_p(\mathbb{R}^n)\| := \Big(\int_{\mathbb{R}^n} |f(x)|^p dx\Big)^{1/p},$$

with the obvious modification if $p = \infty$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) \equiv (\mathcal{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n, \tag{1.5}$$

denotes the Fourier transform of φ. As usual, $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$, stands for the inverse Fourier transform, given by the right-hand side of (1.5) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi(x) = 1 \quad \text{if} \quad |x|_a \le 1 \quad \text{and} \quad \operatorname{supp} \varphi \subset \{x \in \mathbb{R}^n : |x|_a \le 2\}, \tag{1.6}$$

and for each $j \in \mathbb{N}$ let

$$\varphi_j^a(x) := \varphi(2^{-ja}x) - \varphi(2^{(-j+1)a}x), \quad x \in \mathbb{R}^n. \tag{1.7}$$

Then the sequence $(\varphi_j^a)_{j=0}^\infty$, with $\varphi_0 = \varphi$, forms a smooth anisotropic dyadic resolution of unity, cf. [37, Sect. 4.2]. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the compact support of $\varphi_j^a\widehat{f}$ implies by the Paley - Wiener - Schwartz theorem that $(\varphi_j^a\widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1.3.1 *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $a = (a_1, \dots, a_n)$ *an anisotropy, and* $(\varphi_j^a)_{j=0}^\infty$ *a smooth anisotropic dyadic resolution of unity. Then* $B_{pq}^{s,a}(\mathbb{R}^n)$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^n)$ *for which the quasi-norm*

$$\|f \mid B_{pq}^{s,a}(\mathbb{R}^n)\| = \left(\sum_{j=0}^\infty 2^{jsq}\|(\varphi_j^a\widehat{f})^\vee|L_p(\mathbb{R}^n)\|^q\right)^{1/q} \tag{1.8}$$

(*with the usual modification if* $q = \infty$) *is finite.*

Remark 1.3.2 Sometimes the following notation is used $B_{pq}^{\bar{s}}(\mathbb{R}^n)$. Given a space $B_{pq}^{s,a}(\mathbb{R}^n)$ then $\bar{s}$ is calculated by $\bar{s} = \left(\frac{s}{a_1}, \dots, \frac{s}{a_m}\right)$.

Note that there is a parallel definition for spaces of type $F_{pq}^{s,a}(\mathbb{R}^n)$, $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, $a = (a_1, \dots, a_n)$ an anisotropy, when interchanging the order of ℓ_q- and L_p- quasi-norms in (1.8). It is obvious, that the quasi-norm

(1.8) depends on the chosen system $(\varphi_j^a)_{j\in\mathbb{N}_0}$, but not the space $B^{s,a}_{pq}(\mathbb{R}^n)$ (in the sense of equivalent quasi-norms); therefore we omit in our notation the subscript φ in the sequel. It is well-known that $B^{s,a}_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ and $q \geq 1$), and, as in the isotropic case, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s,a}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ for all admissible values of p, q, s, see [45, Sect. 2.3.3]. If $s \in \mathbb{R}$ and $0 < p < \infty$, $0 < q < \infty$ then $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{s,a}_{pq}(\mathbb{R}^n)$, see [57, Sect. 3.5] and [10, Sect. 1.2.10]. Note that we indicated the only (formal) difference to the isotropic counterparts of (1.8) by the additional superscript at the smooth anisotropic dyadic resolution of unity $(\varphi_j^a)_{j=0}^\infty$.

We want to point out that if $0 < p < \infty$ and $s \in \mathbb{R}$ then

$$B^{s,a}_{pp}(\mathbb{R}^n) = F^{s,a}_{pp}(\mathbb{R}^n). \tag{1.9}$$

If $1 < p < \infty$ and $s \in \mathbb{R}$ then (in the sense of equivalent quasi-norms)

$$F^{s,a}_{p2}(\mathbb{R}^n) = H^{s,a}_{p}(\mathbb{R}^n) \tag{1.10}$$

where

$$H^{s,a}_p(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \ \left\| \left(\sum_{k=1}^n (1+\xi_k^2)^{s/(2a_k)} \hat{f} \right)^{\vee} |L_p(\mathbb{R}^n) \right\| < \infty \right\} \tag{1.11}$$

is the anisotropic Bessel potential spaces (see [43, Remark 11], [44, Sect. 2.5.2] and [57, Sect. 3.11]).
Furthermore, if $1 < p < \infty, \ s > 0$ and if $s_1 = s/a_1 \in \mathbb{N}, \ldots, s_n = s/a_n \in \mathbb{N}$ then (in the sense of equivalent quasi-norms)

$$F^{s,a}_{p2}(\mathbb{R}^n) = W^{s,a}_{p}(\mathbb{R}^n) \tag{1.12}$$

where

$$W^{s,a}_p(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \ \|f|L_p(\mathbb{R}^n)\| + \sum_{k=1}^n \left\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \ |L_p(\mathbb{R}^n) \right\| < \infty \right\} \tag{1.13}$$

is the classical anisotropic Sobolev spaces on $\mathbb{R}^n$. As a consequence of (1.9), (1.10) and (1.12) we have

$$B^{s,a}_{22}(\mathbb{R}^n) = F^{s,a}_{22}(\mathbb{R}^n) = H^{s,a}_{2}(\mathbb{R}^n) = W^{s,a}_{2}(\mathbb{R}^n), \tag{1.14}$$

for $s > 0$ and $s_i = s/a_i \in \mathbb{N}, \ i = 1, \ldots, n$.

Remark 1.3.3 *A systematic treatment of the theory of (isotropic)* $B^s_{pq}(\mathbb{R}^n)$ *(and* $F^s_{pq}(\mathbb{R}^n)$*) spaces may be found in the monographs [45], [47], [49] and [50]; see also [11] and [36]. A survey on the basic results for the (anisotropic) spaces* $B^{s,a}_{pq}(\mathbb{R}^n)$ *(and* $F^{s,a}_{pq}(\mathbb{R}^n)$*) is given in [37, Sect. 4.2.1-4.2.4] and [26, Sect. 2.1-2.2]. In addition to the literature mentioned in our introduction we essentially rely on [17] and [18] in the sequel.*

For convenience, in case of $p = q$ we shall stick to the notation

$$B^{s,a}_p(\mathbb{R}^n) = B^{s,a}_{pp}(\mathbb{R}^n) \quad \text{where} \quad 0 < p \leq \infty, \quad s \in \mathbb{R}, \tag{1.15}$$

in the sequel.

2 Properties of anisotropic Besov spaces

The aim of chapter 2 is to prove some new and important properties of anisotropic Besov spaces, which we need later one, and thus to complement results in [7, 17, 37].

2.1 Equivalent norms

We begin this section with the anisotropic counterpart of Proposition 1 in [54].

Proposition 2.1.1 *Let* $0 < p \leq \infty$, $s \in \mathbb{R}$, $a = (a_1, \ldots, a_n)$ *an anisotropy, and* $(\varphi_j^a)_{j=0}^\infty$ *a smooth anisotropic dyadic resolution of unity. Then for each* $f \in B_p^{s,a}(\mathbb{R}^n)$,

$$\left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |(\varphi_j^a \widehat{f})^\vee (2^{-ja} m)|^p \right)^{1/p} \sim \|f | B_p^{s,a}(\mathbb{R}^n)\| \tag{2.1}$$

(*with the usual modification if* $p = \infty$) *where the equivalence constants are independent of* s *and* f.

P r o o f. We want to show that

$$\left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |(\varphi_j^a \widehat{f})^\vee (2^{-ja} m)|^p \right)^{1/p} \sim \|f | B_p^{s,a}(\mathbb{R}^n)\|. \tag{2.2}$$

Taking into account the definition (1.8),

$$\|f | B_p^{s,a}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsp} \|(\varphi_j^a \widehat{f})^\vee | L_p(\mathbb{R}^n)\|^p \right)^{1/p}, \tag{2.3}$$

the assertion reduces to

$$\sum_{m \in \mathbb{Z}^n} |(\varphi_j^a \widehat{f})^\vee (2^{-ja} m)|^p \sim 2^{jn} \left\| (\varphi_j^a \widehat{f})^\vee | L_p(\mathbb{R}^n) \right\|^p \tag{2.4}$$

with equivalence constants independent of $j \in \mathbb{N}_0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Here we use an isotropic result given in [45, Sect. 1.3.3] : adapted to our above notation it states that for $0 < p \leq \infty$ there exist some numbers $\nu_0 > 0$ and $c_2 > c_1 > 0$ such that for all $\nu \geq \nu_0$, and all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with supp $\mathcal{F}\varphi \subset \Omega$ it holds

$$c_1 \sum_{m\in\mathbb{Z}^n} \left|\varphi\left(2^{-\nu}m\right)\right|^p \leq 2^{\nu n} \left\|\varphi|L_p(\mathbb{R}^n)\right\|^p \leq c_2 \sum_{m\in\mathbb{Z}^n} \left|\varphi\left(2^{-\nu}m\right)\right|^p \tag{2.5}$$

(modification if $p = \infty$), where $\Omega \subset \mathbb{R}^n$ is compact. In addition, it is known, cf. [45, Rem. 1.3.3] or [44, Sect. 1.3.5], that if for some suitably chosen $y^0 \in \mathbb{R}^n$ and $b > 0$,

$$\Omega \subset \left\{ y \in \mathbb{R}^n \, : |y_j - y_j^0| \leq b, \; j = 1, \ldots, n \right\}, \tag{2.6}$$

then ν_0 can be taken such that $2^{\nu_0} \sim b$. Thus, with $\varphi = (\varphi_j \widehat{f})^\vee$ this implies $b \sim 2^j$ in the isotropic case, i.e. (2.5) with $\varphi = (\varphi_j \widehat{f})^\vee$ and $\nu = j$ yields the desired (isotropic) result. In order to prove (2.4) we simply modify the above argument slightly : let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\text{supp } \mathcal{F}\psi \subset \Omega^a \subset \left\{ y \in \mathbb{R}^n \, : |y_j - y_j^1| \leq b^{a_j}, \; j = 1, \ldots, n \right\},$$

for suitably chosen $y^1 \in \mathbb{R}^n$ and $b > 0$, then we define

$$\varphi(tx) := \psi\left(t^a x\right), \quad t > 0, \quad x \in \mathbb{R}^n. \tag{2.7}$$

Obviously, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}\varphi(t\xi) = \mathcal{F}\psi\left(t^a\xi\right)$, $t > 0$, $\xi \in \mathbb{R}^n$, and consequently

$$\text{supp } \mathcal{F}\varphi \subset \left\{ y \in \mathbb{R}^n \, : |y_j - y_j^0| \leq b, \; j = 1, \ldots, n \right\}$$

(with $y^0 = b^{1-a}y^1$). Hence application of (2.5) and (2.7) leads to

$$c_1 \sum_{m\in\mathbb{Z}^n} \left|\psi\left(2^{-\nu a}m\right)\right|^p \leq 2^{\nu n} \left\|\varphi|L_p(\mathbb{R}^n)\right\|^p \leq c_2 \sum_{m\in\mathbb{Z}^n} \left|\psi\left(2^{-\nu a}m\right)\right|^p. \tag{2.8}$$

On the other hand, with $a_1 + \cdots + a_n = n$,

$$\begin{aligned} \left\|\varphi|L_p(\mathbb{R}^n)\right\|^p &= \int_{\mathbb{R}^n} |\varphi(x)|^p \, \mathrm{d}x = t^n \int_{\mathbb{R}^n} |\varphi(ty)|^p \, \mathrm{d}y \\ &= t^n \int_{\mathbb{R}^n} |\psi(t^a y)|^p \, \mathrm{d}y = \int_{\mathbb{R}^n} |\psi(z)|^p \, \mathrm{d}z \\ &= \left\|\psi|L_p(\mathbb{R}^n)\right\|^p. \end{aligned} \tag{2.9}$$

Finally, (2.8) and (2.9) with $\psi = (\varphi_j^a \widehat{f})^\vee$, $b \sim 2^j$ and $\nu = j$ finish the proof. □

Next, we present some important equivalent quasi-norms in $B^{s,a}_{pq}$. The theorem below is the anisotropic counterpart of Theorem 2.3.3 of [47, p.98].

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ as in Section 1.3, in particular we have (1.6). We extend the definition of φ^a_j from (1.7) to all integers j. It should be noted that φ^a_0 has now a different meaning as in 1.3, i.e. for $f \in \mathcal{S}'(\mathbb{R}^n)$ then we have that

$$f = (\varphi \hat{f})^\vee + \sum_{j=1}^{\infty} (\varphi^a_j \hat{f})^\vee \quad \text{(convergence in } \mathcal{S}'(\mathbb{R}^n)). \tag{2.10}$$

Let $a_+ = \max(0, a)$ where $a \in \mathbb{R}$ and

$$\sigma_p = n\Big(\frac{1}{p} - 1\Big)_+, \quad 0 < p \leq \infty. \tag{2.11}$$

Theorem 2.1.2 *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s > \sigma_p$ and $a = (a_1, \ldots, a_n)$ an anisotropy, then*

$$\|(\varphi \hat{f})^\vee | L_p(\mathbb{R}^n)\| + \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|(\varphi^a_j \hat{f})^\vee | L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{2.12}$$

and

$$\|f | L_p(\mathbb{R}^n)\| + \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|(\varphi^a_j \hat{f})^\vee | L_p(\mathbb{R}^n)\|^q \right)^{1/q} \tag{2.13}$$

(modification if $q = \infty$) are equivalent quasi-norms in $B^{s,a}_{pq}(\mathbb{R}^n)$.

Proof. We closely follow the proof in [47, Sect. 2.3.3] for the isotropic case.
Step 1. We prove that (2.12) is an equivalent quasi-norm in $B^{s,a}_{pq}(\mathbb{R}^n)$. It is sufficient to show that there exists a constant $c > 0$ such that

$$\|(\varphi^a_j \hat{f})^\vee | L_p(\mathbb{R}^n)\| \leq c 2^{-j\sigma_p} \|(\varphi \hat{f})^\vee | L_p(\mathbb{R}^n)\|, \quad -j \in \mathbb{N}, \tag{2.14}$$

holds, because we need to prove that

$$\left(\sum_{j=-\infty}^{-1} 2^{jsq} \|(\varphi^a_j \hat{f})^\vee | L_p(\mathbb{R}^n)\|^q \right)^{1/q} \leq c \|(\varphi \hat{f})^\vee | L_p(\mathbb{R}^n)\|$$

and this is satisfied if (2.14) is true. For those j's we have that $\varphi^a_j(x) = \varphi^a_j(x)\varphi(x)$ by the support condition (1.6) and (1.7) with $-j \in \mathbb{N}$, and hence

$$\begin{aligned} \|(\varphi^a_j \hat{f})^\vee | L_p(\mathbb{R}^n)\| &= \|(\varphi^a_j ((\varphi \hat{f})^\vee)^\wedge)^\vee | L_p(\mathbb{R}^n)\| \\ &\leq c \|\check{\varphi}^a_j | L_r(\mathbb{R}^n)\| \|(\varphi \hat{f})^\vee | L_p(\mathbb{R}^n)\|, \; r = \min(1, p), \end{aligned} \tag{2.15}$$

where the inequality comes from the Fourier multiplier assertion for entire analytic functions, $\|F^{-1}MFf|L_p(\mathbb{R}^n)\| \leq \|F^{-1}M|L_{\tilde{p}}(\mathbb{R}^n)\|\|f|L_p(\mathbb{R}^n)\|$ where $\tilde{p} = \min(1,p)$, proved in [45, Proposition 1.5.1]. Elementary calculations show that $\check{\varphi}^a_j(x) = 2^{jn}\check{\varphi}_0(2^{ja}x)$ such that $\|\check{\varphi}^a_j|L_r(\mathbb{R}^n)\| = 2^{-j\frac{n}{r}}\|\check{\varphi}_0(2^{ja}\cdot)|L_r(\mathbb{R}^n)\| \leq c2^{-j\frac{n}{r}+jn}$ as $a_1+\cdots+a_n = n$. By (2.15) we thus have that

$$\|(\varphi^a_j\hat{f})^\vee|L_p(\mathbb{R}^n)\| \leq 2^{-jn(\frac{1}{r}-1)}\|(\varphi\hat{f})^\vee|L_p(\mathbb{R}^n)\|$$

and we obtain (2.14) since $\sigma_p = n\left(\frac{1}{r}-1\right)$.

Step 2. We prove that (2.13) is an equivalent quasi-norm in $B^{s,a}_{pq}(\mathbb{R}^n)$. By our assumption $s > \sigma_p$, we may assume that (2.10) converges not only in $\mathcal{S}'(\mathbb{R}^n)$, but also, say almost everywhere in $\mathbb{R}^n$. Then we have

$$\|f|L_p(\mathbb{R}^n)\| \leq c\|(\varphi\hat{f})^\vee|L_p(\mathbb{R}^n)\| + c\left(\sum_{j=1}^\infty \|(\varphi^a_j\hat{f})^\vee|L_p(\mathbb{R}^n)\|^p\right)^{1/p} \tag{2.16}$$

if $0 < p \leq 1$ and a corresponding estimate if $1 < p < \infty$. Now (2.12) and (2.16) prove that (2.13) can be estimated from above by $c\|f|B^{s,a}_{pq}(\mathbb{R}^n)\|$. We consider the converse inequality. Because f is a regular distribution we have a.e. that

$$(\varphi\hat{f})^\vee(x) = f(x)+((1-\varphi(\cdot))\hat{f})^\vee(x) = f(x)+\sum_{j=0}^\infty((1-\varphi(\cdot))\varphi^a_j(\cdot)\hat{f})^\vee(x). \tag{2.17}$$

By the above-mentioned Fourier multiplier assertion we have

$$\|\ (\varphi\hat{f})^\vee|L_p(\mathbb{R}^n)\| \leq c\|f|L_p(\mathbb{R}^n)\| + c\left(\sum_{j=0}^\infty \|(\varphi^a_j\hat{f})^\vee|L_p(\mathbb{R}^n)\|^p\right)^{1/p} \tag{2.18}$$

if $0 < p \leq 1$ and a corresponding estimate if $1 < p < \infty$. Now (2.12) and (2.18) prove that $\|f|B^{s,a}_{pq}(\mathbb{R}^n)\|$ can be estimated from above by the quasi-norm (2.13). □

Remark 2.1.3 *The quasi-norms of type* (2.12), (2.13) *have a continuous counterpart. We introduce* $\rho^a(t\xi) = \varphi(t^a\xi) - \varphi((2t)^a\xi)$ *where* $t > 0$. *Then the counterpart of* (2.12) *reads as follows:*
Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s > \sigma_p$ *and* a *an anisotropy, then*

$$\|f|L_p(\mathbb{R}^n)\| + \left(\int_0^\infty t^{-sq}\|(\rho^a(t\cdot)\hat{f})^\vee|L_p(\mathbb{R}^n)\|^q\frac{dt}{t}\right)^{1/q} \tag{2.19}$$

(modification if $q = \infty$*) is an equivalent quasi-norm in* $B^{s,a}_{pq}(\mathbb{R}^n)$.

2.2 Homogeneity estimate

In following we extend the well-known homogeneity estimate for $B^s_{pq}(\mathbb{R}^n)$

$$\|f(R\cdot)|B^s_{pq}(\mathbb{R}^n)\| \le cR^{s-\frac{n}{p}}\|f|B^s_{pq}(\mathbb{R}^n)\| \quad \text{for all } f\in B^s_{pq}(\mathbb{R}^n),$$

if $R\ge 1$, see [45, Prop. 3.4.1], to anisotropic spaces.

Proposition 2.2.1 *Let $0<p\le\infty$, $0<q\le\infty$, $s>\sigma_p$ and $a=(a_1,\dots,a_n)$ an anisotropy. There exists a constant $c>0$ such that for all $R\ge 1$,*

$$\|f(R\cdot)|B^{s,a}_{pq}(\mathbb{R}^n)\| \le cR^{s-\frac{n}{p}}\|f|B^{s,a}_{pq}(\mathbb{R}^n)\| \quad \textit{for all } f\in B^{s,a}_{pq}(\mathbb{R}^n). \tag{2.20}$$

Proof. We closely follow the proof in [11, Prop. 2.3.1] for the isotropic case. Let $\psi=\varphi_1$ be the same function as in (1.7). We have by (2.19)

$$\|f|L_p(\mathbb{R}^n)\| + \left(\int_0^\infty t^{-sq}\|(\psi(t\cdot)\hat{f})^\vee|L_p(\mathbb{R}^n)\|^q\frac{dt}{t}\right)^{1/q} \tag{2.21}$$

is an equivalent quasi-norm on $B^{s,a}_{pq}(\mathbb{R}^n)$. Elementary calculation shows that

$$\begin{aligned}(\psi(t\cdot)f(R\cdot)^\wedge(\cdot))^\vee(x) &= (\psi(t\cdot)\hat{f}(R^{-1}\cdot))^\vee(x)R^{-n}\\ &= (\psi(t(R\cdot))\hat{f}(\cdot))^\vee(Rx).\end{aligned} \tag{2.22}$$

also in the anisotropic case, where $a_1+\dots+a_n=n$. From (2.21), with $f(Rx)$ in place of $f(x)$, and (2.22) we obtain

$$\begin{aligned}&\|f(R\cdot)|B^{s,a}_{pq}(\mathbb{R}^n)\|\\ &\le c_1\|f(R\cdot)|L_p(\mathbb{R}^n)\| + c_1\left(\int_0^\infty t^{-sq}\|\mathcal{F}^{-1}(\psi(t\cdot)\mathcal{F}[f(R\cdot)])|L_p(\mathbb{R}^n)\|^q\frac{dt}{t}\right)^{1/q}\\ &\le c_2R^{-\frac{n}{p}}\|f|L_p(\mathbb{R}^n)\| + c_3R^{s-\frac{n}{p}}\left(\int_0^\infty t^{-sq}\|\mathcal{F}^{-1}(\psi(t(R\cdot))\mathcal{F}f)|L_p(\mathbb{R}^n)\|^q\frac{dt}{t}\right)^{1/q}\end{aligned}$$

and from here follows (2.20) for $R\ge 1$, $c_1,c_2,c_3>0$ and $s>\sigma_p$. □

2.3 Localisation

The main goal in this section is to extend the localisation property, see [11, Sect. 2.3.2], to anisotropic spaces. First we recall the isotropic case, that we will use in our proof below. Let $x^{j,k}=2^{-j}k$ with $k\in\mathbb{Z}^n$ and $j\in\mathbb{N}$. Let $f\in\mathcal{S}'(\mathbb{R}^n)$ with supp $f\subset Q_b$, where

$$Q_b=\{x\in\mathbb{R}^n: \quad x=(x_1,x_2,\dots,x_n),\ |x_l|<b \quad \text{if} \quad l=1,\dots,n\} \tag{2.23}$$

where $b > 0$ and $b \leq \frac{1}{2}$. Let

$$f^j(x) = \sum_{k\in\mathbb{Z}^n} c_k f(2^{j+1}(x - x^{j,k})), \quad c_k \in \mathbb{C}, \quad j \in \mathbb{N}. \tag{2.24}$$

If $s > \sigma_p,\ 0 < p \leq \infty,\ 0 < d \leq 1/4$ then there exist two constants $c_1, c_2 > 0$ such that for all $f \in B^s_{pp}(\mathbb{R}^n)$

$$c_1\|f^j|B^s_{pp}(\mathbb{R}^n)\| \leq 2^{j(s-\frac{n}{p})}\Big(\sum_{k\in\mathbb{Z}^n}|c_k|^p\Big)^{1/p}\|f|B^s_{pp}(\mathbb{R}^n)\| \leq c_2\|f^j|B^s_{pp}(\mathbb{R}^n)\|. \tag{2.25}$$

Now we can extend the property to anisotropic case. Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ having integer valued components. Let $x^a_{j,k} = 2^{-ja}k$ with $k \in \mathbb{Z}^n$ and $j \in \mathbb{N}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\operatorname{supp} f \subset Q^a_b = \{x \in \mathbb{R}^n : \quad x = (x_1, x_2, \ldots, x_n),\ \ |x|_a < b\} \tag{2.26}$$

where $b > 0$ and $b \leq \frac{1}{4}(2^{a_{\min}} + 1)$. Let

$$f^a_j(x) = \sum_{k\in\mathbb{Z}^n} c_k f(2^{(j+1)a}(x - x^a_{j,k})), \quad c_k \in \mathbb{C}, \quad j \in \mathbb{N} \tag{2.27}$$

where f is a product of one-dimensional functions,

$$f(2^{(j+1)a}(x - x^a_{j,k})) = \prod_{m=1}^{n} f_m(2^{(j+1)a_m}(x_m - 2^{-ja_m}k_m)) \tag{2.28}$$

and $f_1(y) = \cdots = f_n(y)$ where $y \in \mathbb{R}$.

Theorem 2.3.1 *Let $s > \sigma_p$, $0 < p \leq \infty$, $a = (a_1, \ldots, a_n)$ an anisotropy and $0 < b \leq \frac{1}{4}(2^{a_{\min}} + 1)$. There exist two constants $c' > 0$ and $c'' > 0$ such that for all $f \in B^{s,a}_p(\mathbb{R}^n)$ with* supp *$f \subset Q^a_b$ and all $j \in \mathbb{N}$ and all f^a_j given by* (2.27)

$$c'\|f^a_j|B^{s,a}_p(\mathbb{R}^n)\| \leq 2^{j(s-\frac{n}{p})}\Big(\sum_{k\in\mathbb{Z}^n}|c_k|^p\Big)^{1/p}\|f|B^{s,a}_p(\mathbb{R}^n)\| \leq c''\|f^a_j|B^{s,a}_p(\mathbb{R}^n)\|. \tag{2.29}$$

Proof. *Step 1.* At first we prove the left-hand side of (2.29). By (2.27) we have

$$f^a_j(2^{-(j+1)a}x) = \sum_{k\in\mathbb{Z}^n} c_k f(x - 2^a k), \quad c_k \in \mathbb{C}, \quad j \in \mathbb{N}, \tag{2.30}$$

where $f \in B^{s,a}_p(\mathbb{R}^n)$ and (2.26) is true. We would like to show that

$$\Big\| \sum_{k\in\mathbb{Z}^n} c_k f(\cdot - 2^a k)|B^{s,a}_p(\mathbb{R}^n)\Big\| \sim \Big(\sum_{k\in\mathbb{Z}^n} |c_k|^p\Big)^{1/p} \|f|B^{s,a}_p(\mathbb{R}^n)\|. \tag{2.31}$$

We use the characterization of $B^{s,a}_p(\mathbb{R}^n)$ via local means; see [17, Sect. 4.4]. Recall notation (1.2). Let $k \in C^\infty$ so that supp $k \subset B^a = \{y \in \mathbb{R}^n : \ |y|_a \le 1\}$ and

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y) f(x + t^a y)dy, \quad t > 0. \tag{2.32}$$

Let $k_0 \in C^\infty$ such that supp $k_0 \subset B^a$, and $s_1 > \max(s, \sigma_p) + \sigma_p$ then

$$\|f|B^{s,a}_p(\mathbb{R}^n)\| \sim \|k_0(1,f)|L_p(\mathbb{R}^n)\| + \Big(\sum_{j=1}^\infty 2^{jsp}\|k(2^{-j},f)|L_p(\mathbb{R}^n)\|^p\Big)^{1/p}, \tag{2.33}$$

see [17, Sect. 4.4]. We insert (2.30) in (2.32) and obtain

$$\begin{aligned} k\Big(t, \sum_{m\in\mathbb{Z}^n} c_m f(\cdot - 2^a m)\Big)(x) &= \int_{\mathbb{R}^n} k(y)\Big(\sum_{m\in\mathbb{Z}^n} c_m f(x + t^a y - 2^a m)\Big)dy \\ &= \sum_{m\in\mathbb{Z}^n} c_m \int_{\mathbb{R}^n} k(y) f(x - 2^a m + t^a y)dy \\ &= \sum_{m\in\mathbb{Z}^n} c_m k(t,f)(x - 2^a m) \end{aligned} \tag{2.34}$$

and it follows

$$\begin{aligned} \| \sum_{m\in\mathbb{Z}^n} c_m f(\cdot - 2^a m)|B^{s,a}_p(\mathbb{R}^n)\| &\sim \|k_0(1, \sum_{m\in\mathbb{Z}^n} c_m f(\cdot - 2^a m))|L_p(\mathbb{R}^n)\| + \\ + &\ \big(\sum_{j=1}^\infty 2^{jsp}\|k(2^{-j}, \sum_{m\in\mathbb{Z}^n} c_m f(\cdot - 2^a m))|L_p(\mathbb{R}^n)\|^p\big)^{\frac{1}{p}} \\ \sim &\ \Big(\sum_{m\in\mathbb{Z}^n} |c_m|^p\Big)^{1/p}\Big(\|k_0(1,f)|L_p(\mathbb{R}^n)\| + \ \big(\sum_{j=1}^\infty 2^{jsp}\|k(2^{-j},f)|L_p(\mathbb{R}^n)\|^p\big)^{\frac{1}{p}}\Big) \\ \sim &\ \Big(\sum_{m\in\mathbb{Z}^n} |c_m|^p\Big)^{1/p} \|f|B^{s,a}_p(\mathbb{R}^n)\|. \end{aligned}$$

Now the left-hand side of inequality of (2.29) is an easy consequence of Proposition 2.2.1, (2.30) and (2.31):

$$\begin{aligned} \|f_j^a|B_p^{s,a}(\mathbb{R}^n)\| &\leq c2^{j(s-n/p)}\Big\|\sum_{m\in\mathbb{Z}^n} c_m f(\cdot - 2^a m)|B_p^{s,a}(\mathbb{R}^n)\Big\| \\ &\leq c'2^{j(s-n/p)}\Big(\sum_{m\in\mathbb{Z}^n}|c_m|^p\Big)^{1/p}\|f|B_p^{s,a}(\mathbb{R}^n)\|. \qquad (2.35) \end{aligned}$$

Step 2. In this step we prove the right-hand side of (2.29). For this we would like to use the localisation property given in [11, Sect. 2.3.2] if $n = 1$ and for the functions

$$f^{j\alpha}(x) = \sum_{m\in\mathbb{Z}} c_m f(2^{(j+1)\alpha}x - 2^\alpha m), \quad c_m \in \mathbb{C},\ j,\alpha \in \mathbb{N},\ x \in \mathbb{R}\ , \qquad (2.36)$$

where $f \in \mathcal{S}'(\mathbb{R})$. By [11, Sect. 2.3.2/4] we know that there exist two constants $c' > 0$ and $c'' > 0$ such that for all $f \in B^s_{pp}(\mathbb{R})$

$$c'\|f^{j\alpha}|B^s_{pp}(\mathbb{R})\| \leq 2^{j\alpha(s-\frac{1}{p})}\big(\sum_{k\in\mathbb{Z}}|c_k|^p\big)^{1/p}\|f|B^s_{pp}(\mathbb{R})\| \leq c''\|f^{j\alpha}|B^s_{pp}(\mathbb{R})\|, \quad (2.37)$$

as for $n = 1$ isotropic and anisotropic results coincide. For the functions f_j^a given in (2.27) we use the Fubini property of $B_p^{s,a}(\mathbb{R}^n)$, where we use the notation like in remark 1.3.2; see [7, Sect. 6.], i.e.

$$\begin{aligned} &\|f_j^a|B_p^{s,a}(\mathbb{R}^n)\| \sim \\ &\sim \sum_{m=1}^{n}\Big\|\ \|f_j^a(x_1,\ldots,x_{m-1},\cdot,x_{m+1},\ldots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m}|L_p(\mathbb{R}^{n-1})\Big\|_{x'}, \quad (2.38) \end{aligned}$$

where $x' = (x_1,\ldots,x_{m-1},x_{m+1},\ldots,x_n)$ and $s_m = \frac{s}{a_m}$. By (2.27) and (2.28)

$$\begin{aligned} &\|f_j^a(x_1,\ldots,x_{m-1},\cdot,x_{m+1},\ldots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m} = \\ &= \Big\|\sum_{k_m=-\infty}^{\infty} f_m(2^{(j+1)a_m}x_m - 2^{a_m}k_m)\Big[\sum_{\bar{k}\in\mathbb{Z}^{n-1}} c_{(k_1,\ldots,k_n)}\bar{f}\Big]|B_p^{s_m}(\mathbb{R})\Big\|_{x_m}, \quad (2.39) \end{aligned}$$

where $\bar{k} = (k_1,\ldots,k_{m-1},k_{m+1},\ldots,k_n)$ and

$$\begin{aligned} \bar{f} &= f_1(2^{(j+1)a_1}x_1 - 2^{a_1}k_1)\cdots f_{m-1}(2^{(j+1)a_{m-1}}x_{m-1} - 2^{a_{m-1}}k_{m-1}) \times \\ &\quad \times f_{m+1}(2^{(j+1)a_{m+1}}x_{m+1} - 2^{a_{m+1}}k_{m+1})\cdots f_n(2^{(j+1)a_n}x_1 - 2^{a_n}k_n). \quad (2.40) \end{aligned}$$

Let $d_{k_m} = \Big(\sum_{\substack{l\in\mathbb{Z}^n \\ l_m=k_m}} |c_l|^p\Big)^{1/p}$ and without restriction of generality we may assume that $d_{k_m} > 0$; we have that

$$\begin{aligned}
&\|f_j^a(x_1,\dots,x_{m-1},\cdot,x_{m+1},\dots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m} = \\
&= \Big\|\sum_{k_m=-\infty}^{\infty} f_m(2^{(j+1)a_m}x_m - 2^{a_m}k_m)\Big[\sum_{\bar{k}\in\mathbb{Z}^{n-1}} d_{k_m}\frac{c_{(k_1,\dots,k_n)}}{d_{k_m}}\bar{f}\Big]|B_p^{s_m}(\mathbb{R})\Big\|_{x_m}. \qquad (2.41)
\end{aligned}$$

Let $\bar{c_k} = \frac{c_{(k_1,\dots,k_n)}}{d_{k_m}}$ and by (2.41) we get that

$$\begin{aligned}
&\|f_j^a(x_1,\dots,x_{m-1},\cdot,x_{m+1},\dots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m} = \\
&= \Big\|\sum_{k_m=-\infty}^{\infty} d_{k_m}f_m(2^{(j+1)a_m}x_m - 2^{a_m}k_m)\Big[\sum_{\bar{k}\in\mathbb{Z}^{n-1}} \bar{c_k}\bar{f}\Big]|B_p^{s_m}(\mathbb{R})\Big\|_{x_m} \\
&= \Big[\sum_{\bar{k}\in\mathbb{Z}^{n-1}} \bar{c_k}\bar{f}\Big]\Big\|\sum_{k_m=-\infty}^{\infty} d_{k_m}f_m(2^{(j+1)a_m}x_m - 2^{a_m}k_m)|B_p^{s_m}(\mathbb{R})\Big\|_{x_m}. \qquad (2.42)
\end{aligned}$$

By (2.42),

$$\begin{aligned}
&\Big\|\|f_j^a(x_1,\dots,x_{m-1},\cdot,x_{m+1},\dots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m}\ |L_p(\mathbb{R}^{n-1})\Big\|_{x'} = \\
&= \Big\|\Big[\sum_{\bar{k}\in\mathbb{Z}^{n-1}} \bar{c_k}\bar{f}\Big]\ \Big\|\sum_{k_m=-\infty}^{\infty} d_{k_m}f_m(2^{(j+1)a_m}x_m - 2^{a_m}k_m)|B_p^{s_m}(\mathbb{R})\Big\|_{x_m}|L_p(\mathbb{R}^{n-1})\Big\|_{x'} \\
&= \Big\|\sum_{k_m=-\infty}^{\infty} d_{k_m}f_m(2^{(j+1)a_m}x_m - 2^{a_m}k_m)|B_p^{s_m}(\mathbb{R})\Big\|_{x_m} \times \\
&\qquad \times\Big\|\sum_{\bar{k}\in\mathbb{Z}^{n-1}} \bar{c_k}\bar{f}\quad |L_p(\mathbb{R}^{n-1})\Big\|_{x'}. \qquad (2.43)
\end{aligned}$$

Note that

$$\begin{aligned}
\Big\|\sum_{\bar{k}\in\mathbb{Z}^{n-1}} \bar{c_k}\bar{f}\ |L_p(\mathbb{R}^{n-1})\Big\|_{x'} &= \Big(\sum_{\bar{k}\in\mathbb{Z}^{n-1}} |\bar{c_k}|^p\Big)^{1/p} 2^{(j+1)(a_m-n)/p} \times \\
&\times \|f_1\cdots f_{m-1}f_{m+1}\cdots f_n|L_p(\mathbb{R}^{n-1})\|_{x'}, \qquad (2.44)
\end{aligned}$$

recall $-(a_1+\cdots+a_{m-1}+a_{m+1}+\cdots+a_n) = a_m - n$. Now we use (2.37) for the spaces $B_p^{s_m}(\mathbb{R})$ and by (2.43), (2.44)

$$\begin{aligned}
&\Big\| \|f_j^a(x_1,\dots,x_{m-1},\cdot,x_{m+1},\dots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m} \,|L_p(\mathbb{R}^{n-1})\Big\|_{x'} \geq \\
&\geq \; c' 2^{ja_m(s_m-\frac{1}{p})}\Big(\sum_{k_m=-\infty}^{\infty}|d_{k_m}|^p\Big)^{1/p}\|f_m|B_p^{s_m}(\mathbb{R})\|_{x_m}\times \\
&\quad \times\, 2^{\frac{j}{p}(a_m-n)}\Big(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}|\bar{c_k}|^p\Big)^{\frac{1}{p}}\|f_1\cdots f_{m-1}\cdot f_{m+1}\cdots f_n|L_p(\mathbb{R}^{n-1})\|_{x'}. \qquad (2.45)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Big(\sum_{k_m=-\infty}^{\infty}|d_{k_m}|^p\Big)^{1/p} &= \Big(\sum_{k_m=-\infty}^{\infty}\sum_{\substack{l\in\mathbb{Z}^n\\ l_m=k_m}}|c_l|^p\Big)^{1/p} \\
&= \Big(\sum_{l\in\mathbb{Z}^n}^{\infty}|c_l|^p\Big)^{1/p} \qquad (2.46)
\end{aligned}$$

and

$$\begin{aligned}
\Big(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}|\bar{c_k}|^p\Big)^{1/p} &= \Big(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}\frac{|c_{(k_1,\dots,k_n)}|^p}{d_{k_m}^p}\Big)^{1/p} \\
&= \frac{1}{d_{k_m}}\underbrace{\Big(\sum_{\bar{k}\in\mathbb{Z}^{n-1}}|c_{(k_1,\dots,k_n)}|^p\Big)^{1/p}}_{\geq d_{k_m}} \\
&\geq 1. \qquad (2.47)
\end{aligned}$$

By (2.45), (2.46) and (2.47) and $s_m\cdot a_m = s$, we conclude

$$\begin{aligned}
&\Big\| \|f_j^a(x_1,\dots,x_{m-1},\cdot,x_{m+1},\dots,x_n)|B_p^{s_m}(\mathbb{R})\|_{x_m} \,|L_p(\mathbb{R}^{n-1})\Big\|_{x'} \geq \\
&\geq \; c' 2^{j(s-\frac{n}{p})}\Big(\sum_{k\in\mathbb{Z}^n}|c_k|^p\Big)^{1/p}\|f_m|B_p^{s_m}(\mathbb{R})\|_{x_m}\times \\
&\quad \times\|f_1\cdots f_{m-1}\cdot f_{m+1}\cdots f_n|L_p(\mathbb{R}^{n-1})\|_{x'} \\
&\geq \; c' 2^{j(s-\frac{n}{p})}\Big(\sum_{k\in\mathbb{Z}^n}|c_k|^p\Big)^{1/p}\Big\|\|f_1\cdots f_n|B_p^{s_m}(\mathbb{R})\|_{x_m}|L_p(\mathbb{R}^{n-1})\Big\|_{x'}. \qquad (2.48)
\end{aligned}$$

By (2.48) and the Fubini property (2.38) we obtain the right-hand side of inequality of (2.29)

$$\begin{aligned}\|f_j^a|B_p^{s,a}(\mathbb{R}^n)\| &\geq c2^{j(s-\frac{n}{p})}\left(\sum_{k\in\mathbb{Z}^n}|c_k|^p\right)^{1/p}\sum_{m=1}^{n}\Big\|\|f_1\cdots f_n|B_p^{s_m}(\mathbb{R})\|_{x_m}|L_p(\mathbb{R}^{n-1})\Big\|_{x'} \\ &\geq c'2^{j(s-\frac{n}{p})}\left(\sum_{k\in\mathbb{Z}^n}|c_k|^p\right)^{1/p}\|f|B_p^{s,a}(\mathbb{R}^n)\|. \qquad (2.49)\end{aligned}$$

□

3 Decompositions in anisotropic Besov spaces

Many procedures were established to reduce problems in function spaces to the level of sequence spaces with the help of decomposition techniques. There are many different possible ways to do so, for example by using molecules, atoms, quarks and wavelets. In this chapter we give sub-atomic and wavelet representations of anisotropic Besov spaces. The arguments stressed there are essentially based on (known) corresponding atomic decomposition. Thus we recall some basic facts about anisotropic atoms and atomic decomposition in section 3.1 and in section 3.2 we formulate our main results.

3.1 Anisotropic atoms and the atomic decomposition theorem

Let $a = (a_1, \ldots, a_n)$ an anisotropy, $\nu \in \mathbb{N}_0$, and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, then we denote by $Q^a_{\nu m}$ the rectangle in $\mathbb{R}^n$ centered at $2^{-\nu a}m = (2^{-\nu a_1}m_1, \ldots$ $\ldots, 2^{-\nu a_n}m_n)$ which has sides parallel to axes and side lengths respectively $2^{-\nu a_1}, \ldots, 2^{-\nu a_n}$. Note that Q^a_{0m} is a cube with side length 1. If $Q^a_{\nu m}$ is such a rectangle in $\mathbb{R}^n$ and $c > 0$ then $cQ^a_{\nu m}$ is the rectangle in $\mathbb{R}^n$ concentric with $Q^a_{\nu m}$ and with side lengths respectively $c2^{-\nu a_1}, \ldots, c2^{-\nu a_n}$.

Definition 3.1.1

(i) *Let $K \in \mathbb{R}$, $c > 1$. A function $\varrho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\gamma \varrho$ if $a\gamma \le K$ (continuous if $K \le 0$), is called an* ***anisotropic*** 1_K-***atom*** *if*

$$\operatorname{supp} \varrho \subset cQ^a_{0m} \quad \textit{for some } m \in \mathbb{Z}^n, \tag{3.1}$$

$$|D^\gamma \varrho(x)| \le 1 \quad \textit{if} \quad a\gamma \le K, \quad \gamma \in \mathbb{N}^n_0,\ x \in \mathbb{R}^n. \tag{3.2}$$

(ii) *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K, L \in \mathbb{R}$. A function $\varrho : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\gamma \varrho$ if $a\gamma \le K$ (continuous if $K \le 0$) is called an* ***anisotropic*** $(s,p)_{K,L}-$***atom*** *if*

$$\operatorname{supp} \varrho \subset cQ^a_{\nu m} \quad \textit{for some} \quad \nu \in \mathbb{N},\ m \in \mathbb{Z}^n, \tag{3.3}$$

$$|D^\gamma \varrho(x)| \le 2^{-\nu(s-\frac{n}{p})+\nu a\gamma} \quad \textit{if} \quad a\gamma \le K, \quad \gamma \in \mathbb{N}_0^n, \quad x \in \mathbb{R}^n, \tag{3.4}$$

$$\int_{\mathbb{R}^n} x^\beta \varrho(x)\,\mathrm{d}x = 0 \quad \textit{if} \quad a\beta \le L, \quad \beta \in \mathbb{N}_0^n. \tag{3.5}$$

If the atom ϱ is located at $Q^a_{\nu m}$ (that means supp $\varrho \subset cQ^a_{\nu m}$ with $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $c > 1$) then we shall denote it by $\varrho^a_{\nu m}$.

Remark 3.1.2 The value of the number $c > 1$ in (3.1) and (3.3) is unimportant; it only indicates that at level ν some controlled overlapping of the supports of $\varrho^a_{\nu m}$ has to be allowed. The moment conditions (3.5) can be reformulated as

$$D^\beta \widehat{\varrho}(0) = 0 \quad \text{if} \quad a\beta \le L,$$

which shows that a sufficiently strong decay of $\widehat{\varrho}$ at the origin is required. If $L < 0$ then (3.5) should be interpreted in the sense that there are no moment conditions. The normalising factors in (3.2), (3.4) imply that there exists a constant $c > 0$ such that for all these atoms we have $\|\varrho \,|\, B^{s,a}_{pq}(\mathbb{R}^n)\| \le c$, see Theorem 3.1.4 below. Hence, as in the isotropic case, atoms are normalised building blocks satisfying some moment conditions.

This construction generalises isotropic atoms leading back to [19], [20] and the survey [21]. It is also slightly related to the concept of anisotropic building blocks (compactly supported and satisfying some norming and some moment conditions) used in [41] to define anisotropic Hardy spaces and to study the relation of these spaces to anisotropic Lipschitz and Campanato - Morrey spaces. As already mentioned, we use the presentation from [17], which itself was motivated by the isotropic counterparts in [47], [49].

Suitable anisotropic sequence spaces can be introduced as follows.

Definition 3.1.3 *Let $0 < p \le \infty$, $0 < q \le \infty$, and $a = (a_1, \ldots, a_n)$ an anisotropy. Then b_{pq} is the collection of all sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$ such that*

$$\|\lambda | b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} \tag{3.6}$$

(*with the usual modification if $p = \infty$ and/or $q = \infty$*) *is finite.*

Again, note that there is a counterpart for spaces of type $F^{s,a}_{pq}(\mathbb{R}^n)$; the corresponding sequence spaces f^a_{pq} can be introduced similarly, but will not be used in the sequel. One can easily check that b_{pq} are quasi-Banach spaces. For $0 < p \le \infty$ let σ_p given by (2.11).

Theorem 3.1.4 *[17, Thm.3.3] Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and let $K, L \in \mathbb{R}$ be such that*

$$K \geq a_{\max} + s \quad if \quad s \geq 0, \tag{3.7}$$

$$L \geq \sigma_p - s. \tag{3.8}$$

Then $g \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{s,a}_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$g = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} \varrho^a_{\nu m}, \tag{3.9}$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where $\varrho^a_{\nu m}$ are anisotropic $1_K -$ atoms $(\nu = 0)$ or anisotropic $(s,p)_{K,L} -$ atoms $(\nu \in \mathbb{N})$ and $\lambda \in b_{pq}$ with $\lambda = \{\lambda_{\nu m} : \ \nu \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\}$. Furthermore,

$$\inf \|\lambda | b_{pq}\|$$

where the infimum is taken over all admissible representations (3.9)*, is an equivalent quasi-norm in $B^{s,a}_{pq}(\mathbb{R}^n)$.*

Remark 3.1.5 A proof of this theorem – and its counterpart for spaces $F^{s,a}_{pq}(\mathbb{R}^n)$ – is given in [17, Sect. 5.1]. The convergence in $\mathcal{S}'(\mathbb{R}^n)$ can be obtained as a by-product of the proof using the same method as for its isotropic counterpart in [49, Sect. 13.9]. As already mentioned it generalises atomic decomposition results in [19], [20], [49], to anisotropic function spaces.

3.2 Decompositions and wavelets

Our main object is to study the anisotropic counterpart of results from [52]; hence we closely follow this presentation, adapting it to our context when necessary, but keeping similar notation if possible. Let

$$\mathbb{R}^n_{++} = \{y \in \mathbb{R}^n : y = (y_1, \ldots, y_n), \ y_j > 0 \quad \text{for all } j\} \tag{3.10}$$

and let k be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset \{y \in \mathbb{R}^n : |y|_a < 2^J\} \cap \mathbb{R}^n_{++}, \tag{3.11}$$

for some $J \in \mathbb{N}$, and

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1, \quad x \in \mathbb{R}^n. \tag{3.12}$$

Recall $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\beta \in \mathbb{N}^n_0$, and put

$$k^\beta(x) = (2^{-Ja} x)^\beta k(x) \geq 0, \quad x \in \mathbb{R}^n, \ \beta \in \mathbb{N}^n_0. \tag{3.13}$$

Let

$$\lambda = \{\lambda^\beta_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \beta \in \mathbb{N}^n_0\}. \tag{3.14}$$

For $s \in \mathbb{R},\ 0 < p \leq \infty$, and $\varrho \geq 0$, we introduce $b^{s,\varrho}_p$ by

$$\|\lambda|b^{s,\varrho}_p\| = \left(\sum_{\beta\in\mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{\varrho a\beta p + j(s-n/p)p} |\lambda^\beta_{jm}|^p \right)^{1/p}. \tag{3.15}$$

Let

$$\omega \in \mathcal{S},\ \text{supp}\ \omega \subset (-\pi,\pi)^n,\ \ \omega(x) = 1 \quad \text{if} \quad |x|_a \leq 2, \tag{3.16}$$

and let

$$\omega^\beta(x) = \frac{i^{|\beta|} 2^{Ja\beta}}{(2\pi)^n \beta!} x^\beta \omega(x) \qquad \text{for } x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}^n_0, \tag{3.17}$$

recall (1.1). Let

$$\Omega^\beta(x) = \sum_{m\in\mathbb{Z}^n} (\omega^\beta)^\vee(m) e^{-imx}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}^n_0. \tag{3.18}$$

Definition 3.2.1 *Let φ_0 be a C^∞ function in $\mathbb{R}^n$ with*

$$\varphi_0(x) = 1 \quad \textit{if} \quad |x|_a \leq 1 \quad \textit{and} \quad \varphi_0(x) = 0 \quad \textit{if} \quad |x|_a \geq \frac{3}{2}, \tag{3.19}$$

*and let $\varphi(x) = \varphi_0(x) - \varphi_0(2^a x)$ and $\beta \in \mathbb{N}^n_0$. The **father wavelets** $\Phi^\beta_F(x)$ and the **mother wavelets** $\Phi^\beta_M(x)$ are given by*

$$\left(\Phi^\beta_F\right)^\vee(\xi) = \varphi_0(\xi)\Omega^\beta(\xi), \ \ \xi \in \mathbb{R}^n, \tag{3.20}$$

$$\left(\Phi^\beta_M\right)^\vee(\xi) = \varphi(\xi)\Omega^\beta(\xi), \ \ \xi \in \mathbb{R}^n. \tag{3.21}$$

Remark 3.2.2 Our assumption $\omega^\beta \in \mathcal{S}(\mathbb{R}^n)$ implies that $\left(\Phi^\beta_F\right)^\vee$, $\left(\Phi^\beta_M\right)^\vee$, and hence also Φ^β_F, Φ^β_M are elements of $\mathcal{S}(\mathbb{R}^n)$. Furthermore, Φ^β_F and Φ^β_M are entire analytic functions with vanishing moments of arbitrary order for Φ^β_M, because (3.19) implies supp $\varphi \subset \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x|_a \leq \frac{3}{2}\}$, and thus (3.21) yields $D^\alpha \left(\Phi^\beta_M\right)^\vee(0) = 0,\ \ \alpha \in \mathbb{N}^n_0$, which can be reformulated as

$$\int_{\mathbb{R}^n} \Phi^\beta_M(\xi)\xi^\alpha \,\mathrm{d}\xi = 0, \quad \alpha \in \mathbb{N}^n_0. \tag{3.22}$$

By construction we have

$$\Phi_F^\beta(x) = \sum_{m\in\mathbb{Z}^n} (\omega^\beta)^\vee(m)\widehat{\varphi_0}(x+m), \quad x \in \mathbb{R}^n, \tag{3.23}$$

$$\Phi_M^\beta(x) = \sum_{m\in\mathbb{Z}^n} (\omega^\beta)^\vee(m)\widehat{\varphi}(x+m), \quad x \in \mathbb{R}^n. \tag{3.24}$$

For fundamentals about wavelets we refer, for instance, to [32] and [56].

Let Φ_F^β and Φ_M^β, $\beta \in \mathbb{N}_0^n$, be given by Definition 3.2.1, and introduce for $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, the wavelets

$$\Phi_{jm}^\beta(x) = \begin{cases} \Phi_F^\beta(x-m), & \text{if } j = 0, \\ \Phi_M^\beta(2^{ja}x - m), & \text{if } j \in \mathbb{N}. \end{cases} \tag{3.25}$$

According to the dual pairing $(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ we put, for given $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\lambda_{jm}^\beta(f) = 2^{jn}(\Phi_{jm}^\beta, f), \;\; j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n, \; \beta \in \mathbb{N}_0^n. \tag{3.26}$$

Finally, let

$$B_p^{+,a}(\mathbb{R}^n) = \bigcup_{s>0} B_p^{s,a}(\mathbb{R}^n), \;\; 0 < p \le \infty.$$

Recall our notation (2.11).

Theorem 3.2.3 *Let $0 < p \le \infty$, $s > \sigma_p$, $\varrho \ge 0$, and $a = (a_1, \ldots, a_n)$ an anisotropy.*

(i) *Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is an element of $B_p^{s,a}(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta\in\mathbb{N}_0^n} \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{jm}^\beta \; k^\beta(2^{ja}x - m), \quad x \in \mathbb{R}^n, \tag{3.27}$$

with $\|\lambda|b_p^{s,\varrho}\| < \infty$, absolute convergence being in $L_{\max(1,p)}$ if $\max(1,p) < \infty$ and if $p = \infty$ in $L_\infty(\mathbb{R}^n, w)$ with $w(x) = (1+|x|^2)^{\sigma/2}$ where $\sigma < 0$. Furthermore,

$$\|f|B_p^{s,a}(\mathbb{R}^n)\| \sim \inf \; \|\lambda|b_p^{s,\varrho}\|, \tag{3.28}$$

where the infimum is taken over all admissible representations (3.27).

(ii) *Let* $\lambda^{\beta}_{jm}(f)$ *be given by* (3.26). *Then* $f \in B^{+,a}_p(\mathbb{R}^n)$ *can be represented as*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^{\beta}_{jm}(f)\; k^{\beta}(2^{ja}x - m), \tag{3.29}$$

absolute convergence being in $L_{\max(1,p)}$ *if* $\max(1,p) < \infty$ *and if* $p = \infty$ *in* $L_\infty(\mathbb{R}^n, w)$ *with* $w(x) = (1 + |x|^2)^{\sigma/2}$ *where* $\sigma < 0$, *and, in addition,* $f \in B^{s,a}_p(\mathbb{R}^n)$ *if, and only if,* $\|\lambda(f)|b^{s,\varrho}_p\| < \infty$.

(iii) *Let* $f \in B^{s,a}_p(\mathbb{R}^n)$, *then* (3.29) *is an optimal representation, i.e.*

$$\|f|B^{s,a}_p(\mathbb{R}^n)\| \sim \|\lambda(f)|b^{s,\varrho}_p\|, \tag{3.30}$$

where the equivalence constants are independent of f.

Proof. <u>*Step 1.*</u> We assume that f is given by (3.27) with $\|\lambda|b^{s,\varrho}_p\| < \infty$ for some $\varrho \geq 0$. We want to show that $f \in B^{s,a}_p(\mathbb{R}^n)$ and that there exists a constant $c > 0$ such that

$$\|f|B^{s,a}_p(\mathbb{R}^n)\| \leq c\|\lambda|b^{s,\varrho}_p\|. \tag{3.31}$$

We rewrite (3.27) as

$$f = \sum_{\beta} f^{\beta} \quad \text{with} \quad f^{\beta} = \sum_{j,m} \lambda^{\beta}_{j,m} k^{\beta}(2^{ja}x - m). \tag{3.32}$$

By definition (3.11), the support of k is contained in an open ball centered at the origin and of radius $2^{J-\varepsilon}$ for some $\varepsilon > 0$. Using the atomic decomposition for the spaces $B^{s,a}_p(\mathbb{R}^n)$ with $0 < p \leq \infty$ and $s > \sigma_p$ described in Theorem 3.1.4, we find by Definition 3.1.1 that

$$\{2^{\varepsilon a\beta} 2^{-j(s-n/p)} k^{\beta}(2^{ja}x - m) : \;\; j \in \mathbb{N}_0, \;\; m \in \mathbb{Z}^n\}, \;\; \beta \in \mathbb{N}_0^n, \tag{3.33}$$

are admitted systems of anisotropic atoms, and hence $f^{\beta} \in B^{s,a}_p(\mathbb{R}^n)$ for all $\beta \in \mathbb{N}_0^n$, with

$$\big\|f^{\beta}|B^{s,a}_p(\mathbb{R}^n)\big\| \leq \; c\, 2^{-\varepsilon a\beta} \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |\lambda^{\beta}_{jm}|^p \right)^{1/p}, \tag{3.34}$$

and c independent of $\beta \in \mathbb{N}_0^n$. Summation over β proves $f \in B^{s,a}_p(\mathbb{R}^n)$ and (3.31). The absolute convergence follows in the same way as discussed in [52] and [50, Sect. 1.4, 2.7].

Step 2. Let $f \in B_p^{s,a}(\mathbb{R}^n)$ with $0 < p \leq \infty$ and $s > \sigma_p$; we shall show that we can decompose it as (3.29). Note that this covers then (i) as well.

Let R_j^a, $j \in \mathbb{N}_0$, be a rectangle in $\mathbb{R}^n$ centered at the origin with side-length $2\pi 2^{ja}$ where $2^{ja} = (2^{ja_1}, \ldots, 2^{ja_n})$. Let φ_j^a be given by (1.7), that is, with $\operatorname{supp} \varphi_j^a \subset R_j^a$. Now we can write that

$$\widehat{f}(x) = \sum_{j=0}^{\infty} \varphi_j^a(x)\widehat{f}(x), \quad x \in \mathbb{R}^n,$$

and expand $\varphi_j^a \widehat{f}$ in R_j^a into a Fourier series,

$$(\varphi_j^a \widehat{f})(\xi) = \sum_{m \in \mathbb{Z}^n} b_{jm} \exp(-i2^{-ja} m\xi), \quad \xi \in R_j^a. \tag{3.35}$$

We calculate b_{jm}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, by

$$\begin{aligned}
(\varphi_j^a \widehat{f})^\vee(2^{-ja}m) &= (2\pi)^{-\frac{n}{2}} \int_{R_j^a} (\varphi_j^a \widehat{f})(y) \exp(i2^{-ja}my)\, \mathrm{d}y \\
&= (2\pi)^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} b_{jk} \int_{R_j^a} \exp\left(i2^{-ja}(m-k)y\right)\, \mathrm{d}y \\
&= (2\pi)^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} b_{jk} \int_{R_j^a} \exp\left(i \sum_{l=1}^{n} 2^{-ja_l}(m_l - k_l)y_l\right) \mathrm{d}y.
\end{aligned}$$

Substitute $\xi_l = 2^{-ja_l} y_l$, then $\mathrm{d}\xi = 2^{-j(a_1+\cdots+a_n)}\, \mathrm{d}y = 2^{-jn}\, \mathrm{d}y$, and we arrive at

$$(\varphi_j^a \widehat{f})^\vee(2^{-ja}m) = 2^{jn}\, (2\pi)^{-\frac{n}{2}} \sum_{k \in \mathbb{Z}^n} b_{jk} \int_{Q_\pi} e^{i(m-k)\xi}\, \mathrm{d}\xi, \tag{3.36}$$

where Q_π is a cube of side-length 2π in each direction. For the latter term in (3.36) we thus have

$$\int_{Q_\pi} e^{i(m-k)\xi}\, \mathrm{d}\xi = \left\{ \begin{array}{ccc} (2\pi)^n & , & m = k \\ 0 & , & m \neq k \end{array} \right\},$$

which by (3.36) finally leads to

$$\begin{aligned}
b_{jm} &= (2\pi)^{-\frac{n}{2}}\, 2^{-jn}\, (\varphi_j^a \widehat{f})^\vee(2^{-ja}m) \\
&= (2\pi)^{-n}\, 2^{-jn} \int_{R_j^a} (\varphi_j^a \widehat{f})(\xi)\, \exp(i2^{-ja}m\xi)\, \mathrm{d}\xi.
\end{aligned} \tag{3.37}$$

By Proposition 2.1.1 and (3.37) we thus have for $0 < p \le \infty$,

$$\|f|B_p^{s,a}(\mathbb{R}^n)\| \sim \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)p}\ 2^{jnp} \sum_{m\in\mathbb{Z}^n} |b_{jm}|^p\right)^{1/p}, \tag{3.38}$$

(with the usual modification if $p = \infty$). Let ω be given by (3.16) and let $\omega_j(x) = \omega(2^{-ja}x)$. The ω_j has a compact support in R_j^a and it follows by (3.35) that

$$\begin{aligned}(\varphi_j^a \widehat{f})^\vee(x) &= \sum_{m\in\mathbb{Z}^n} b_{jm}\ \omega_j^\vee(x - 2^{-ja}m) \\ &= 2^{jn} \sum_{m\in\mathbb{Z}^n} b_{jm}\ \omega^\vee(2^{ja}x - m), \quad x \in \mathbb{R}^n. \end{aligned}\tag{3.39}$$

Let k be given by (3.11), (3.12). We expand the analytic function $\omega^\vee(2^{ja}x-m)$ at $2^{-ja}l$, $l \in \mathbb{Z}^n$, and obtain

$$\begin{aligned} k(2^{ja}x-l)\omega^\vee(2^{ja}x-m) & \\ &= \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{ja\beta}}{\beta!}(D^\beta\omega^\vee)(l-m)(x-2^{-ja}l)^\beta k(2^{ja}x-l) \\ &= \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{ja\beta}}{\beta!}(D^\beta\omega^\vee)(l-m)2^{-ja\beta}(2^{ja}x-l)^\beta k(2^{ja}x-l) \\ &= \sum_{\beta\in\mathbb{N}_0^n} \frac{2^{Ja\beta}}{\beta!}(D^\beta\omega^\vee)(l-m)k^\beta(2^{ja}x-l), \end{aligned}\tag{3.40}$$

where we applied (3.13) in the last line. By (3.12), (3.39) and (3.40) we obtain

$$\begin{aligned}(\varphi_j^a \widehat{f})^\vee(x) &= \sum_{m\in\mathbb{Z}^n} 2^{jn} b_{jm} \sum_{l\in\mathbb{Z}^n} k(2^{ja}x-l)\omega^\vee(2^{ja}x-m) \\ &= \sum_{\beta\in\mathbb{N}_0^n} \sum_{l\in\mathbb{Z}^n} k^\beta(2^{ja}x-l) \sum_{m\in\mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!}(D^\beta\omega^\vee)(l-m). \end{aligned}$$

Hence, as $\left(\varphi_j^a\right)_{j\in\mathbb{N}_0}$ is a resolution of unity,

$$f = \sum_{j=0}^{\infty} \sum_{\beta\in\mathbb{N}_0^n} \sum_{l\in\mathbb{Z}^n} k^\beta(2^{ja}x-l)\ \lambda_{jl}^\beta \tag{3.41}$$

with

$$\lambda_{jl}^\beta = \sum_{m\in\mathbb{Z}^n} 2^{jn}\ b_{jm}\ \frac{2^{Ja\beta}}{\beta!}\ (D^\beta\omega^\vee)(l-m). \tag{3.42}$$

We first check that those λ^β_{jl} are optimal coefficients, and verify their representation (3.26) afterwards. We thus claim that for $\varrho \geq 0$ we can find a constant c such that with $\lambda = \{\lambda^\beta_{jl} : \beta \in \mathbb{N}^n_0, j \in \mathbb{N}_0, l \in \mathbb{Z}^n\}$ given by (3.42),

$$\|\lambda | b^{s,\varrho}_p\| \leq c \|f | B^{s,a}_p(\mathbb{R}^n)\| \quad \text{for all} \quad f \in B^{s,a}_p(\mathbb{R}^n). \tag{3.43}$$

We use an isotropic result [51, Sect. 3.1.1], which states that for any $\varepsilon > 0$ there are constants $c > 0$ and $c_\varepsilon > 0$ such that

$$|D^\beta \omega^\vee(x)| \leq \; c_\varepsilon \; 2^{c|\beta|} \; (1 + |x|^2)^{-\varepsilon} \quad \text{for} \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}^n_0, \tag{3.44}$$

where c is independent of x, ε, and β, and c_ε independent of x, β. Furthermore, note that there are constants $c_2 > c_1 > 0$ such that for all $\xi \in \mathbb{R}^n$,

$$c_1 \, (1 + |\xi|)^{1/a_{\max}} \leq \; 1 + |\xi|_a \leq \; c_2 \, (1 + |\xi|)^{1/a_{\min}} \, ,$$

cf. [31]. On the other hand, we have $a_{\min}|\beta| \leq a\beta \leq a_{\max}|\beta|$, thus (3.44) implies

$$|D^\beta \omega^\vee(x)| \leq \; c'_\varepsilon \; 2^{c' a\beta} \; (1 + |x|_a)^{-\varepsilon} \quad \text{for} \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}^n_0. \tag{3.45}$$

Let $p \geq 1$. We interprete λ^β_{jl} as a convolution in ℓ_p : let $l \in \mathbb{Z}^n$ and

$$a_l = \sum_{m \in \mathbb{Z}^n} c_m d_{l-m},$$

then $\|a_l | \ell_p\| \leq \|d_k | \ell_1\| \|c_m | \ell_p\|$. Put $d_k = \dfrac{2^{Ja\beta}}{\beta!} \, (D^\beta \omega^\vee)(k)$, then (3.45) leads to

$$\|d_k | \ell_1\| \; \leq \; C_\varepsilon \; \frac{2^{(J+c')a\beta}}{\beta!} \; \leq \; c(\varrho) \; 2^{-(\varrho+1)a\beta},$$

if $\varepsilon > 0$ is chosen appropriately. The last inequality results from an estimate of $\beta! = \beta_1! \cdots \beta_n!$ by Stirling's formula, $n! = \Gamma(n+1) \sim \sqrt{n} \left(\frac{n}{e}\right)^n$, $n \in \mathbb{N}$. Consequently, (3.42) with $a_l = \lambda^\beta_{jl}$ and $c_m = 2^{jn} \; b_{jm}$ implies, that for $\varrho \geq 0$ there is a constant $c(\varrho)$ such that

$$\left(\sum_{l \in \mathbb{Z}^n} |\lambda^\beta_{jl}|^p \right)^{1/p} \leq c(\varrho) 2^{-(\varrho+1)a\beta} \left(\sum_{l \in \mathbb{Z}^n} |2^{jn} b_{jl}|^p \right)^{1/p}. \tag{3.46}$$

If $p < 1$ then one uses the $p-$triangle inequality. Now (3.43) follows from (3.46) and (3.38).

Step 3. We need to prove that λ_{jl}^β can be represented as (3.26). By (3.37) and the properties of the Fourier transform we have

$$2^{jn} b_{jm} = (2\pi)^{-n} \int_{\mathbb{R}^n} (\varphi_j^a)^\vee (2^{-ja} m - y) f(y)\, \mathrm{d}y, \quad j \in \mathbb{N}_0, \tag{3.47}$$

where φ is given by Definition 3.2.1. Now $\varphi_j^a(x) = \varphi(2^{-ja}x)$ leads to

$$2^{jn} b_{jm} = (2\pi)^{-n} 2^{jn} \int_{\mathbb{R}^n} \varphi^\vee (m - 2^{-ja} y) f(y)\, \mathrm{d}y, \quad j \in \mathbb{N}. \tag{3.48}$$

Recall that $(D^\beta \omega^\vee)(\xi) = i^{|\beta|}(x^\beta \omega(x))^\vee(\xi)$. Thus (3.17) implies for $j \in \mathbb{N}$,

$$\begin{aligned} \lambda_{jl}^\beta &= \sum_{m \in \mathbb{Z}^n} 2^{jn} b_{jm} \frac{2^{Ja\beta}}{\beta!} (D^\beta \omega^\vee)(l-m) \\ &= \sum_{m \in \mathbb{Z}^n} 2^{jn}\ b_{jm}\ \frac{2^{Ja\beta}}{\beta!}\ i^{|\beta|}\ (x^\beta \omega(x))^\vee (l-m) \\ &= \sum_{m \in \mathbb{Z}^n} 2^{jn}\ b_{jm}\ \frac{2^{Ja\beta}}{\beta!}\ i^{|\beta|}\ \frac{(2\pi)^n \cdot \beta!}{i^{|\beta|} 2^{Ja\beta}}\ (\omega^\beta)^\vee (l-m) \\ &= 2^{jn} \int_{\mathbb{R}^n} f(y) \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee (l-m)\ \varphi^\vee (m - 2^{ja} y)\, \mathrm{d}y. \end{aligned} \tag{3.49}$$

Replacing $l-m$ by m and using $\varphi^\vee(z) = \widehat{\varphi}(-z)$ we get

$$\begin{aligned} \lambda_{jl}^\beta &= 2^{jn} \int_{\mathbb{R}^n} f(y) \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee (m)\ \widehat{\varphi}(2^{ja} y - l + m)\, \mathrm{d}y \\ &= 2^{jn} (\Phi_{jl}^\beta, f), \quad j \in \mathbb{N}, \end{aligned} \tag{3.50}$$

where we used (3.24) and (3.25). The argument for $j = 0$ works analogously, so the proof is finished. □

Remark 3.2.4 The isotropic version of this result can be found in [52] which in turn is a specification and modification of [50, Thm. 2.9], where also further references and approaches are discussed. Note that (i) represents a so-called sub-atomic (or quarkonial) decomposition in $B_p^{s,a}(\mathbb{R}^n)$; we refer also to [17, Thm. 3.7].

We study the "dual" situation, i.e. spaces $B_p^{s,a}(\mathbb{R}^n)$ with $s < 0$, too. For that purpose, denote the counterpart of $B_p^{+,a}(\mathbb{R}^n)$ by

$$B_p^{-,a}(\mathbb{R}^n) = \bigcup_{s<0} B_p^{s,a}(\mathbb{R}^n), \quad 0 < p \le \infty. \tag{3.51}$$

Let k and k^β be given by (3.12) and (3.13), and consider the corresponding local means,

$$k^\beta(t,f)(x) = \int_{\mathbb{R}^n} k^\beta(y) f(x + t^a y)\,\mathrm{d}y, \quad t > 0, \quad x \in \mathbb{R}^n, \tag{3.52}$$

where $x + t^a y = (x_1 + t^{a_1} y_1, \ldots, x_n + t^{a_n} y_n)$, and put

$$k^\beta_{jm}(f) = k^\beta(2^{-j}, f)(2^{-ja} m), \ \ j \in \mathbb{N}_0, \ \ m \in \mathbb{Z}^n. \tag{3.53}$$

We use the norm given by (3.15) with $\varrho = 0$, denoted simply by $b^s_p = b^{s,0}_p$ for convenience. Let $k(f) = \{k^\beta_{jm}(f) : j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \ \beta \in \mathbb{N}^n_0\}$, hence

$$\|k(f)|b^s_p\| = \Big(\sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{j(s-n/p)p} |k^\beta_{jm}(f)|^p \Big)^{\frac{1}{p}}, \tag{3.54}$$

and let Φ^β_{jm} defined in (3.25).

Theorem 3.2.5 *Let* $1 < p \le \infty, \ s < 0$.

(i) *Then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *is an element of* $B^{s,a}_p(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\beta_{jm} \Phi^\beta_{jm}, \ \ x \in \mathbb{R}^n, \tag{3.55}$$

with $\|\lambda|b^s_p\| < \infty$, *and unconditional convergence in* $\mathcal{S}'(\mathbb{R}^n)$. *Furthermore,*

$$\|f|B^{s,a}_p(\mathbb{R}^n)\| \sim \inf \ \|\lambda|b^s_p\|, \tag{3.56}$$

where the infimum is taken over all admissible representations (3.55).

(ii) *Any* $f \in B^{-,a}_p(\mathbb{R}^n)$ *can be represented as*

$$f = \sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} k^\beta_{jm}(f) \ \Phi^\beta_{jm}, \tag{3.57}$$

unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and, in addition,* $f \in B^{s,a}_p(\mathbb{R}^n)$ *if, and only if,* $\|k(f)|b^s_p\| < \infty$.

(iii) *Let* $f \in B^{s,a}_p(\mathbb{R}^n)$, *then* (3.57) *is an optimal representation, i.e.*

$$\|f|B^{s,a}_p(\mathbb{R}^n)\| \sim \|k(f)|b^s_p\|, \tag{3.58}$$

where the equivalence constants are independent of f.

We begin the proof of Theorem 3.2.5 with some preparation. Let $l \in \mathbb{Z}^n$, $K,\ L \in \mathbb{R}$, and $a_l \in C^K$ anisotropic atoms given by Definition 3.1.1 with $\operatorname{supp} a_l \subset \{y \in \mathbb{R}^n :\ |y|_a \le c\}$, for some appropriate $c > 0$. We know by (3.5) that

$$\int_{\mathbb{R}^n} x^\beta a_l(x)\,\mathrm{d}x = 0 \quad \text{if} \quad a\beta \le L. \tag{3.59}$$

Let $\mu = (\mu_l)_{l\in\mathbb{Z}^n}$ denote the decay factors,

$$|D^\gamma a_l(x)| \le \mu_l, \quad a\gamma \le K, \quad x \in \mathbb{R}^n. \tag{3.60}$$

We define now (special) **anisotropic molecules**

$$b(x) = \sum_{l\in\mathbb{Z}^n} a_l(x-l), \quad x \in \mathbb{R}^n, \tag{3.61}$$

and

$$b_{j,m}(x) = 2^{-j(s-\frac{n}{p})}\ b(2^{ja}x - m), \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ x \in \mathbb{R}^n. \tag{3.62}$$

Remark 3.2.6 Normalised (isotropic) molecules share the decay properties and moment conditions with normalised (isotropic) atoms, but lack the assumption concerning the compact support, see (the isotropic counterparts of) (3.1), (3.3). This is replaced by sufficiently strong decay assumptions. In the isotropic case the counterpart of Theorem 3.1.4 remains valid if one uses molecular instead of atomic decompositions, cf. [21, Sect. 5]. There are also anisotropic versions of that result in the literature, see [6]. However, by the special structure of building blocks we shall use, we do not need this assertion in its full generality, but only a special case which is simpler to prove. Thus we include this consideration below separately and give a direct proof.

Proposition 3.2.7 *Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$, and $\mu = (\mu_l)_{l\in\mathbb{Z}^n} \in \ell_{\min(1,p)}$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be given by*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\ b_{j,m}, \tag{3.63}$$

where $\lambda = \{\lambda_{j,m} \in \mathbb{C} :\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$, $\lambda \in b_{pq}$, and $b_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are given by (3.62)*. Then*

$$\|f|B^{s,a}_{pq}(\mathbb{R}^n)\| \le\ c\ \|\lambda|b_{pq}\|. \tag{3.64}$$

P r o o f. We thank the idea to this proof some discussions with Prof. H. Triebel.

By definition and (3.61), (3.62),

$$\begin{aligned} f &= \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\ b_{j,m} \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m}\ b(2^{ja}x-m) \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} \sum_{l\in\mathbb{Z}^n} a_l(2^{ja}x-m-l) \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{k\in\mathbb{Z}^n} \sum_{l\in\mathbb{Z}^n} \lambda_{j,k-l}\ a_l(2^{ja}x-k) \\ &= \sum_{j=0}^{\infty} 2^{-j(s-\frac{n}{p})} \sum_{k\in\mathbb{Z}^n} \gamma_{j,k}\ d_{j,k}(x) \end{aligned} \tag{3.65}$$

where we have put

$$\gamma_{j,k} = \sum_{l\in\mathbb{Z}^n} \mu_l |\lambda_{j,k-l}|\ > 0, \quad j\in\mathbb{N}_0,\ k\in\mathbb{Z}^n,$$

and

$$d_{j,k}(x) = \gamma_{j,k}^{-1} \sum_{l\in\mathbb{Z}^n} \lambda_{j,k-l}\ a_l(2^{ja}x-k), \quad j\in\mathbb{N}_0,\ k\in\mathbb{Z}^n,\ x\in\mathbb{R}^n. \tag{3.66}$$

We claim that $2^{-j(s-\frac{n}{p})}\ d_{j,k}$ are anisotropic atoms according to Definition 3.1.1. Assume first $j=0$, then

$$|d_{0,k}(x)| \le \gamma_{0,k}^{-1} \sum_{l\in\mathbb{Z}^n} |\lambda_{0,k-l}| \cdot \mu_l = 1,$$

and likewise for all derivatives $D^\gamma d_{0,k}$, $\gamma\in\mathbb{N}_0^n$, $a\gamma\le K$, according to (3.60). In case of $j\in\mathbb{N}$ and $\gamma\in\mathbb{N}_0^n$ we conclude similarly,

$$\begin{aligned} \left|D^\gamma\left(2^{-j(s-\frac{n}{p})}\ d_{j,k}(x)\right)\right| &\le 2^{-j(s-\frac{n}{p})}\ \gamma_{j,k}^{-1} \sum_{l\in\mathbb{Z}^n} |\lambda_{j,k-l}|\ \left|(D^\gamma a_l)\,(2^{ja}x-k)\right| 2^{ja\gamma} \\ &\le 2^{-j(s-\frac{n}{p})+ja\gamma}\ \gamma_{j,k}^{-1} \sum_{l\in\mathbb{Z}^n} |\lambda_{j,k-l}|\ \mu_l \\ &= 2^{-j(s-\frac{n}{p})+ja\gamma} \end{aligned}$$

as long as $a\gamma \le K$. The corresponding moment conditions (3.5) are satisfied by (3.59), and condition (3.3) is guaranteed by construction (3.66). Hence $2^{-j(s-\frac{n}{p})}\, d_{j,k}$ are anisotropic atoms and Theorem 3.1.4 gives the anisotropic atomic decomposition

$$f = \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} \gamma_{j,k}\ 2^{-j(s-\frac{n}{p})}\ d_{j,k}$$

with

$$\left\| f|B^{s,a}_{pq}(\mathbb{R}^n)\right\| \le\ c\ \|\gamma|b_{pq}\|\,.$$

It remains to estimate $\|\gamma|b_{pq}\|$ by $\|\lambda|b_{pq}\|$, where the assumption on the decay factors, i.e. $\mu \in \ell_{\min(p,1)}$ is now involved. Let first $p \ge 1$, then we interprete $\gamma_{j,k}$ as a convolution in ℓ_p and obtain

$$\left(\sum_{k\in\mathbb{Z}^n} |\gamma_{j,k}|^p\right)^{\frac{1}{p}} \le \left(\sum_{l\in\mathbb{Z}^n} \mu_l\right) \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\right)^{\frac{1}{p}} \ =\ \|\mu|\ell_1\| \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p\right)^{\frac{1}{p}}.$$

If $0 < p < 1$, we have by the $p-$triangle inequality

$$\sum_{k\in\mathbb{Z}^n} |\gamma_{j,k}|^p \ \le\ \sum_{k,l\in\mathbb{Z}^n} \mu_l^p\ |\lambda_{j,k-l}|^p \ \le\ \|\mu|\ell_p\|^p \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|^p.$$

This finally results in

$$\|\gamma|b_{pq}\| \le\ c\left\|\mu|\ell_{\min(1,p)}\right\|\ \|\lambda|b_{pq}\|$$

as desired, i.e. $\left\|f|B^{s,a}_{pq}(\mathbb{R}^n)\right\| \le\ c\,\|\gamma|b_{pq}\| \le\ c'\left\|\mu|\ell_{\min(1,p)}\right\| \|\lambda|b_{pq}\|.$ □

We finally can prove the Theorem 3.2.5.

P r o o f.(of Theorem 3.2.5)

Step 1. We assume that f is given by (3.55) with $\|\lambda|b^s_p\| < \infty$. We want to show that $f \in B^{s,a}_p(\mathbb{R}^n)$ and that there exists a constant $c > 0$ such that

$$\|f|B^{s,a}_p(\mathbb{R}^n)\| \le c\|\lambda|b^s_p\|. \tag{3.67}$$

Recall our particular construction (3.23), (3.24), (3.25) with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ given by Definition 3.2.1. Then for each $\beta \in \mathbb{N}^n_0$,

$$\{2^{\varepsilon a\beta}2^{-j(s-\frac{n}{p})}\tilde{\Phi}^{\beta}_{jm}:\quad j\in\mathbb{N}_0,\quad m\in\mathbb{Z}^n\} \tag{3.68}$$

where $\tilde{\Phi}^{\beta}_{jm} = \sum_{l\in\mathbb{Z}^n} k(x-l)\Phi^{\beta}_{jm}$, are admitted anisotropic special molecules in $B^{s,a}_p(\mathbb{R}^n)$ in the above sense (3.59)-(3.62), satisfying, in addition, the decay assumption $\mu \in \ell_{\min(p,1)}$. Thus we can apply Proposition 3.2.7 with $q=p$ and then (3.67) follows in the same way as in *Step 1* in the proof of Theorem 3.2.3. This covers the unconditional convergence, too.

Step 2. Let $k(f)=\{k^\beta_{jm}(f)\ : \beta\in\mathbb{N}_0^n,\ j\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\}$ be given by (3.52), (3.53), i.e.

$$\begin{aligned} k^\beta_{jm}(f) &= k^\beta\left(2^{-j},f\right)\left(2^{-ja}m\right)\\ &= \int_{\mathbb{R}^n} k^\beta(y)\ f\left(2^{-ja}m+2^{-ja}y\right)\ \mathrm{d}y\\ &= 2^{jn}\int_{\mathbb{R}^n} k^\beta(2^{ja}y-m)f(y)\,\mathrm{d}y\\ &= 2^{jn}\ \left(k^\beta(2^{ja}\cdot-m),f\right). \end{aligned} \tag{3.69}$$

We have to show that there is some $c>0$ such that for all $f\in B^{s,a}_p(\mathbb{R}^n)$,

$$\|k(f)|b^s_p\| \le c\ \left\|f|B^{s,a}_p(\mathbb{R}^n)\right\|. \tag{3.70}$$

Using (3.54) with $\varrho=0$ (recall our convention $b^s_p=b^{s,0}_p$), (3.69) can thus be rewritten as

$$\begin{aligned} \left\|k(f)|b^s_p\right\| &= \left\|\sum_{\beta,j,m} 2^{j(s-\frac{n}{p})}k^\beta_{jm}\Big|\ell_p\right\|\\ &= \left\|\sum_{\beta,j,m} 2^{j(s-\frac{n}{p})+jn}\left(k^\beta(2^{ja}\cdot-m),f\right)\Big|\ell_p\right\|. \end{aligned} \tag{3.71}$$

By duality, $\ell_p=(\ell_{p'})'$ with $\frac{1}{p}+\frac{1}{p'}=1$ for $1<p\le\infty$, taking additionally $2^{j(s-\frac{n}{p})+jn}=2^{j(s+\frac{n}{p'})}$ into account, we are thus led to

$$\begin{aligned} \left\|k(f)|b^s_p\right\| &= \left\|\sum_{\beta,j,m} 2^{j(s+\frac{n}{p'})}\left(k^\beta(2^{ja}\cdot-m),f\right)\Big|\ell_p\right\|\\ &= \sup_{\left\|\lambda|b^{-s}_{p'}\right\|\le 1}\ \sum_{\beta,j,m}\lambda^\beta_{jm}\left(k^\beta(2^{ja}\cdot-m),f\right), \end{aligned} \tag{3.72}$$

where the supremum in (3.72) is taken over all sequences $\lambda=\{\lambda^\beta_{jm}\ :\ \beta\in\mathbb{N}_0^n,\ j\in\mathbb{N}_0,\ m\in\mathbb{Z}^n\}$, such that the right-hand side in (3.72) is non-negative, and $\|\lambda|b^{-s}_{p'}\|\le\ 1$. Consequently,

$$\left\|k(f)|b^s_p\right\| \le \sup_{\left\|\lambda|b^{-s}_{p'}\right\|\le 1} |(g,f)|,\quad \text{where}\quad g(x)=\sum_{\beta,j,m}\lambda^\beta_{jm}k^\beta(2^{ja}x-m). \tag{3.73}$$

Note that our assumptions $s < 0$ and $1 < p \le \infty$ imply $-s > 0 = \sigma_{p'}$ such that Theorem 3.2.3 can be applied to $g \in B^{-s,a}_{p'}(\mathbb{R}^n)$. Thus we arrive at

$$\left\|k(f)|b^s_p\right\| \le \sup\left\{|(g,f)| \;: g \in B^{-s,a}_{p'}(\mathbb{R}^n),\ \|g|B^{-s,a}_{p'}(\mathbb{R}^n)\| \le c\right\}, \qquad (3.74)$$

where $c > 0$ is independent of $g \in B^{-s,a}_{p'}(\mathbb{R}^n)$. Now we use the duality

$$(B^{-\sigma,a}_{p'}(\mathbb{R}^n))' = B^{\sigma,a}_{p}(\mathbb{R}^n), \quad 1 \le p' < \infty, \quad \sigma \in \mathbb{R}, \qquad (3.75)$$

see [53], and obtain

$$\left\|k(f)|b^s_p\right\| \le c' \left\|f|B^{s,a}_p(\mathbb{R}^n)\right\|. \qquad (3.76)$$

Step 3. Let $1 < p \le \infty$, $s < 0$ and $f \in B^{s,a}_p(\mathbb{R}^n)$. It is sufficient to verify (3.57) in order to complete the proof. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be arbitrary, then Theorem 3.2.3, in particular (3.29) with (3.26), yields

$$\psi = \sum_{\beta,j,m} 2^{jn} (\Phi^\beta_{jm}, \psi)\; k^\beta(2^{ja}x - m) \qquad (3.77)$$

with unconditional convergence in any space $B^{\sigma,a}_{p'}(\mathbb{R}^n)$ with $\sigma > 0$. Hence, by (3.69),

$$(f,\psi) = \left(\sum_{\beta,j,m} 2^{jn}\left(f, k^\beta(2^{ja}\cdot - m)\right)\Phi^\beta_{jm}, \psi\right) = \left(\sum_{\beta,j,m} k^\beta_{jm}(f)\Phi^\beta_{jm}, \psi\right), \quad (3.78)$$

that is

$$f = \sum_{\beta,j,m} k^\beta_{jm}(f)\; \Phi^\beta_{jm}$$

in $\mathcal{S}'(\mathbb{R}^n)$. By (i) and our preceding remarks it follows that (3.57) converges unconditionally in $\mathcal{S}'(\mathbb{R}^n)$. □

Remark 3.2.8 Parallel to Remark 3.2.4 we refer to the isotropic version of the above result in [52] with further discussions (about local means) in [47] and [50].

4 Traces and approximation numbers

As an application of the wavelet decomposition theorem given in section 3.2 in this chapter we give a unified approach to the study of traces on anisotropic function spaces. In section 4.1 we study the existence and properties of the trace operator. In section 4.2 we recall the concept of approximation numbers. In section 4.3 we define anisotropic $d-$sets and in the last section we obtain estimates for the approximation numbers of traces on anisotropic $d-$sets from $\mathbb{R}^n$.

4.1 General measures

Let μ be a positive Radon measure in $\mathbb{R}^n$ with compact support

$$\Gamma = \text{supp}\ \mu, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{4.1}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. For $1 \leq p < \infty$ we denote by $L_p(\Gamma) = L_p(\Gamma, \mu)$ the usual complex Banach space, normed by

$$\|f|L_p(\Gamma,\mu)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mu(dx) \right)^{1/p} = \left(\int_{\Gamma} |f(\gamma)|^p \mu(d\gamma) \right)^{1/p}.$$

Since μ is Radon, $S(\mathbb{R}^n)|\Gamma$ is dense in $L_p(\Gamma)$, for details see [49, p.7]. If $\varphi \in \mathcal{S}$ then $tr_\Gamma \varphi = \varphi|\Gamma$ makes sense pointwise. If $1 < p < \infty$ and $s > 0$ then the embedding $tr_\Gamma B^{s,a}_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)$ must be understood as follows: we ask whether there is a positive number $c > 0$ such that for any $\varphi \in \mathcal{S}$,

$$\|tr_\Gamma \varphi|L_p(\Gamma)\| \leq c\|\varphi|B^{s,a}_p(\mathbb{R}^n)\|. \tag{4.2}$$

If this is the case we use that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^{s,a}_p(\mathbb{R}^n)$ for $0 < p < \infty$ this inequality can be extended by completion to any $f \in B^{s,a}_p(\mathbb{R}^n)$ and the resulting function is denoted by $tr_\Gamma f$

$$tr_\Gamma : B^{s,a}_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma) \tag{4.3}$$

and the independence of $tr_\Gamma f$ from the approximating sequence is shown in the standard way. On the other hand, if $f \in L_p(\Gamma)$ is given, then f can be interpreted in the usual way as a tempered distribution $id_\Gamma f$, given by

$$\begin{aligned}(id_\Gamma f)(\varphi) &= \int_\Gamma f(\gamma)\varphi(\gamma)\mu(d\gamma) \\ &= \int_\Gamma f(\gamma)(tr_\Gamma \varphi)(\gamma)\mu(d\gamma), \quad \varphi \in \mathcal{S}(\mathbb{R}^n).\end{aligned} \tag{4.4}$$

We call id_Γ the identification operator. Let again $1 < p < \infty$ and let

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{4.5}$$

Then

$$(L_p(\Gamma))' = L_{p'}(\Gamma) \quad \text{and} \quad (B_p^{\sigma,a}(\mathbb{R}^n))' = B_{p'}^{-\sigma,a}(\mathbb{R}^n) \tag{4.6}$$

for any $\sigma \in \mathbb{R}$. The first assertion is well known, the second it follows by [55, Sect. 5.1.7]. In particular, all $B_p^{s,a}(\mathbb{R}^n)$ and also $L_p(\Gamma)$ with $1 < p < \infty$ are reflexive. By (4.4), the operators tr_Γ and id_Γ are dual to each other. Hence, (4.3) is equivalent to

$$id_\Gamma : \; L_{p'}(\Gamma) \hookrightarrow B_{p'}^{-s,a}(\mathbb{R}^n), \tag{4.7}$$

and

$$tr_\Gamma^{'} = id_\Gamma \quad id_\Gamma^{'} = tr_\Gamma. \tag{4.8}$$

In following we study the existence of the trace operator. We proceed similar to [54], dealing with the isotropic case. Let Q_{jm}^a be the rectangles in $\mathbb{R}^n$ with side length $2^{-ja_1}, \ldots, 2^{-ja_n}$ and centered at $2^{-ja}m$ where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. Let

$$\mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}^a), \quad j \in \mathbb{N}_0. \tag{4.9}$$

Proposition 4.1.1 *Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad s > 0.$$

Let μ be the Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \text{supp } \mu \quad \textit{compact}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{4.10}$$

and

$$\sum_{j \in \mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} < \infty \quad \textit{where} \quad \mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}^a). \tag{4.11}$$

Then tr_Γ,

$$tr_\Gamma : B^{s,a}_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma) \tag{4.12}$$

exists and is compact. Furthermore there is a constant c (depending on p and s) such that for all measures μ *with* (4.10), (4.11),

$$\|tr_\Gamma\| \le c\Big(\sum_{j\in\mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})}\mu_j^{p'-1}\Big)^{\frac{1}{p'}}. \tag{4.13}$$

Remark 4.1.2 The result above is the anisotropic version of [54, Proposition 3]. In our proof below we use the wavelet decomposition introduced in Theorem 3.2.3. In the sequel we shall stick to the notation

$$k^\beta_{jm}(x) = k^\beta(2^{ja}x - m), \quad \beta \in \mathbb{N}^n_0, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n. \tag{4.14}$$

P r o o f.(of Proposition 4.1.1)
Step 1. We prove the existence of (4.12) like in the isotropic case, see [54, Prop. 2], [50, Theorem 9.3, Corollary 9.8]. In our case we use the anisotropic local means and the equivalent norm in anisotropic function spaces, see [18, Sect. 2.2]. In comparison with [50, Theorem 9.3] we need only a special case where $u = v = p'$ and $\sigma = -s$. On the other hand, the existence of tr_Γ can also be shown by similar arguments as presented below.
Step 2. Let $f \in B^{s,a}_p(\mathbb{R}^n)$ be given by (3.29), (3.30) (we use the notation (4.14)). For any fixed $\beta \in \mathbb{N}^n_0$ we have

$$\begin{aligned}\Big\|\sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{jm}(f)k^\beta_{jm}|L_p(\Gamma)\Big\| &\le \sum_{j=0}^\infty \Big\|\sum_{m\in\mathbb{Z}^n}\lambda^\beta_{jm}(f)k^\beta_{jm}|L_p(\Gamma)\Big\| \\ &\le c\sum_{j=0}^\infty \Big(\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}|\lambda^\beta_{jm}(f)|^p|k^\beta_{jm}(x)|^p\mu(dx)\Big)^{1/p} \\ &\le c\sum_{j=0}^\infty\Big(\sum_{m\in\mathbb{Z}^n}|\lambda^\beta_{jm}(f)|^p\int_{cQ^a_{jm}}|k^\beta_{jm}(x)|^p\mu(dx)\Big)^{1/p} \\ &\le c'\sum_{j=0}^\infty \mu_j^{\frac{1}{p}}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda^\beta_{jm}(f)|^p\Big)^{1/p} \end{aligned} \tag{4.15}$$

where we used the boundedness of k and (4.9). We apply the Hölder inequality, recall $\frac{1}{p}+\frac{1}{p'}=1$, and so we can continue

$$\begin{aligned}&\Big\|\sum_{j=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda^\beta_{jm}(f)k^\beta_{jm}|L_p(\Gamma)\Big\| \\ &\le c'\Big(\sum_{j=0}^\infty 2^{-jp'(s-\frac{n}{p})}\mu_j^{\frac{p'}{p}}\Big)^{1/p'}\Big(\sum_{j,m}2^{j(s-\frac{n}{p})p}|\lambda^\beta_{jm}(f)|^p\Big)^{1/p}. \end{aligned} \tag{4.16}$$

We choose $\varrho > 0$. Then it follows by (3.15) and (3.30) that

$$\|tr_\Gamma f|L_p(\Gamma)\| \le c'\left(\sum_{j=0}^{\infty} 2^{-jp'(s-\frac{n}{p})}\mu_j^{p'-1}\right)^{1/p'} \|f|B_p^{s,a}(\mathbb{R}^n)\| \tag{4.17}$$

where c' is independent of μ. This proves (4.13).

Step 3. We prove that tr_Γ is compact. Let $B \in \mathbb{N}$, $J \in \mathbb{N}$, $[a\beta] = \max\{r \in \mathbb{Z} : \; r \le a\beta\}$, and let $tr_\Gamma^{B,J}$ be given by

$$tr_\Gamma^{B,J} f = \sum_{[a\beta]\le B} \sum_{j\le J} \sum_{m\in\mathbb{Z}^n}^{\Gamma} \lambda_{jm}^\beta(f) k_{jm}^\beta, \tag{4.18}$$

where again $f \in B_p^{s,a}(\mathbb{R}^n)$ is given by (3.29),(3.30) and where the sum $\sum_{m\in\mathbb{Z}^n}^{\Gamma}$ is restricted to those $m \in \mathbb{Z}^n$ such that the rectangles Q_{jm}^a have a non-empty intersection with Γ. For given $\delta > 0$ and suitably chosen $\varrho > 0$ it follows by the above arguments for $f \in B_p^{s,a}(\mathbb{R}^n)$ having norm of at most 1 that

$$\begin{aligned}\|(tr_\Gamma - tr_\Gamma^{B,J}) f|L_p(\Gamma)\| &\le \\ \le \; & c\left(\sum_{[a\beta]\ge B} 2^{-\delta a\beta}\right) + c\left(\sum_{[a\beta]\le B} 2^{-\delta a\beta}\right)\left(\sum_{j\ge J} 2^{-jp'(s-\frac{n}{p})}\mu_j^{p'-1}\right)^{1/p'},\end{aligned} \tag{4.19}$$

see (3.15) and (4.16), (4.17). By (4.11) we find for any given $\varepsilon > 0$ sufficiently large numbers B and J such that

$$\|tr_\Gamma - tr_\Gamma^{B,J}\| \le \varepsilon. \tag{4.20}$$

Then tr_Γ is compact, as $tr_\Gamma^{B,J}$ are finite rank operators. □

4.2 Approximation numbers

In the following we recall the concept of approximation numbers. Let A and B be two quasi Banach spaces. The family of all linear bounded operators $T : \; A \to B$ will be denoted by $L(A,B)$ or $L(A)$ if $A = B$. Let $T \in L(A,B)$, then for any $k \in \mathbb{N}$ the **kth approximation number** $a_k(T)$ of T is given by

$$a_k(T) = \inf\{\|T - L\| \;\; : \; L \in L(A,B), \quad \text{rank } L < k\}, \tag{4.21}$$

where rank L is the dimension of the range of L. These numbers have various properties given in the following lemma.

Lemma 4.2.1 *Let A, B, C be Banach spaces, let $T, S \in L(A, B)$ and suppose that $R \in L(B, C)$.*

(i) $\|T\| = a_1(T) \geq a_2(T) \cdots \geq 0$

(ii) *for all* $n,\ m \in \mathbb{N}$,

$$a_{m+n-1}(S + T) \leq a_m(S) + a_n(T)$$

(iii) *for all* $n,\ m \in \mathbb{N}$,

$$a_{m+n-1}(R \circ T) \leq a_m(R)a_n(T)$$

(iv) $a_n(T) = 0 \Longleftrightarrow \text{rank}\, T < n$.

These formulations coincide essentially with [13, II. Prop. 2.2], where one finds also a short proof. Further properties, comments and references to the literature may be found in [11, p.11-18], [13, Chap. II.] and [8]. We restrict ourselves to those assertions which we need later on.
Let A be a complex quasi-Banach space and $T \in L(A)$ a compact map. We know from [11, Theorem 1.2] that the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\lambda_k(T) : \ k \in \mathbb{N}\}$ be the sequence of all nonzero eigenvalues of T, repeated according to algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \ldots \geq 0. \tag{4.22}$$

If T has only $m(< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_k(T) = 0$ for $k > M$.

Proposition 4.2.2 *(i) Let A and B two Banach spaces and $T \in L(A, B)$ with dual operator $T' \in L(A', B')$, then*

$$a_k(T) = a_k(T') \quad \textit{for all} \quad k \in \mathbb{N}. \tag{4.23}$$

(ii) Let H be a Hilbert space and let $T \in L(H)$ be a compact, non-negative and self adjoint operator. Then the approximation numbers $a_k(T)$ of T coincide with its eigenvalues (ordered as in (4.22)*).*

Remark 4.2.3 Proofs of these well-known assertions may be found in [13], Proposition 2.5, p.55 for (i), and Theorem 5.10, p.91 for (ii).

At the end of this chapter we compare our results with Farkas results for the entropy number, see [18, Sect. 6.1], and we recall an important connection between entropy numbers and approximation numbers.
Let $0 < p < \infty$, A and B be an arbitrary Banach spaces and $T \in L(A, B)$. Then

$$\sup_{k=1,\dots,m} k^{\frac{1}{p}}\, e_k(T) \le c \sup_{k=1,\dots,m} k^{\frac{1}{p}}\, a_k(T) \tag{4.24}$$

where $c = c(p) > 0$, see [8, Theorem 3.1.1].

In following we estimate the approximation numbers of the compact trace operator. Let $T = tr_\Gamma$ according to Proposition 4.1.1. We strengthen (4.11) by

$$\sum_{j \ge J} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} \sim 2^{-Jp'(s-\frac{n}{p})} \mu_J^{p'-1}, \quad J \in \mathbb{N}_0, \tag{4.25}$$

where only the cases $s \le \frac{n}{p}$ are of interest, otherwise (4.25) is always satisfied.

Proposition 4.2.4 *Let*

$$1 < p < \infty, \;\; \frac{1}{p} + \frac{1}{p'} = 1, \;\; s > 0.$$

Let μ be a Radon measure in $\mathbb{R}^n$ with (4.10) *and* (4.25). *Let $a_k = a_k(tr_\Gamma)$ be the approximation numbers of the compact operator tr_Γ in* (4.12). *There are two positive numbers c and c' such that*

$$a_{c2^{nJ}} \le c' 2^{-J(s-\frac{n}{p})} \mu_J^{\frac{1}{p}}, \;\; J \in \mathbb{N}_0, \tag{4.26}$$

where $c2^{nJ}$ is always assumed to be a natural number.

Proof. Note that (4.25) implies (4.11), thus by Proposition 4.1.1 the operator tr_Γ is compact. We refine (4.18) by

$$tr_\Gamma^J f = \sum_{[a\beta] \le J} \sum_{j \le J-[a\beta]} \sum_{m \in \mathbb{Z}^n}^{\Gamma} \lambda_{jm}^\beta(f) k_{jm}^\beta, \quad J \in \mathbb{N}, \tag{4.27}$$

where again $f \in B_p^{s,a}(\mathbb{R}^n)$ is given by (3.29),(3.30) and the last sum has the same meaning as the last sum in (4.18). As μ is a measure in $\mathbb{R}^n$ we have that

$$\mu_K \le c 2^{(J-K)n} \mu_J, \quad K \le J, \tag{4.28}$$

also in the anisotropic case, recall $a_1 + \dots + a_n = n$. Let $\delta > 0$ be sufficiently large. By (4.28) we obtain for $f \in B_p^{s,a}(\mathbb{R}^n)$ having norm of at most 1 in analogy to (4.19) that

$$\begin{aligned}
\|(tr_\Gamma - tr_\Gamma^J)f|L_p(\Gamma)\| &\leq c2^{-\delta J} + c\sum_{[a\beta]\leq J} 2^{-\delta a\beta}\Bigg(\sum_{j\geq J-[a\beta]} 2^{-jp'(s-\frac{n}{p})}\mu_j^{\frac{p'}{p}}\Bigg)^{1/p'} \\
&\leq c2^{-\delta J} + c\sum_{[a\beta]\leq J} 2^{-\delta a\beta}2^{-(J-[a\beta])(s-\frac{n}{p})}\mu_{J-[a\beta]}^{\frac{1}{p}} \\
&\leq c2^{-\delta J} + c\mu_J^{\frac{1}{p}}2^{-J(s-\frac{n}{p})}\sum_{[a\beta]\leq J} 2^{-\delta a\beta + a\beta(s-\frac{n}{p})+a\beta\frac{n}{p}} \\
&\leq c'2^{-J(s-\frac{n}{p})}\mu_J^{\frac{1}{p}}. \qquad (4.29)
\end{aligned}$$

In the second estimate we used assumption (4.25) and in the next one (4.28). For the rank of tr_Γ^J we have the estimate

$$\text{rank}(tr_\Gamma^J) \leq c\sum_{[a\beta]\leq J} 2^{n(J-[a\beta])} \leq c'2^{nJ}.$$

This proves (4.26). □

4.3 Anisotropic $d-$sets in $\mathbb{R}^n$

We consider special measures μ and assume $\Gamma = \text{supp}\ \mu$ for some measure according to Section 4.1, in particular with (4.1), now. Let again $a = (a_1, \ldots, a_n)$ be a given anisotropy.

Definition 4.3.1 *Let* $0 < d < n$. *Then* $\Gamma \subset \mathbb{R}^n$ *is called an* **anisotropic** $d-$**set** *if*

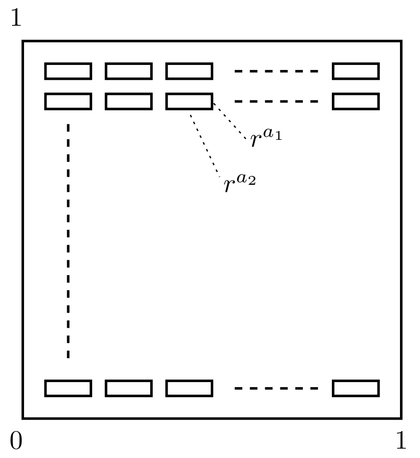
$$\mu(B^a(\gamma, r)) \sim r^d, \quad 0 < r < 1, \qquad (4.30)$$

where $B^a(\gamma, r) = \{y \in \mathbb{R}^n : |y-\gamma|_a \leq r\}$ *and* $\gamma \in \Gamma$.

In the following proposition we prove the existence of anisotropic $d-$sets.

Proposition 4.3.2 *For every* $0 < d < n$ *there exists an anisotropic* $d-$*set.*

P r o o f. For simplicity we prove this proposition for the case $n = 2$. If $n > 2$ this can be done in a similar way.



We use the well-known mass distribution procedure to construct a measure μ with the desired properties. We refer to [49, Ch.4] for details. Let $Q = [0,1]^2$ be the closed cube with side-length 1, we take the affine contractions $(A_m)_{m=1}^N$ on $\mathbb{R}^2$ which map the unit square to the rectangles $(A_mQ)_{m=1}^N$ with side-lengths r^{a_1} and r^{a_2} where $0 < a_1 < a_2$ and $a_1 + a_2 = 2$ as in Figure 1, so that they are disjoint and $\mu(A_mQ) = N^{-1}$. Furthermore we have $Nr^2 = N|A_mQ| < 1$. Let

$$AQ = (AQ)^1 = \bigcup_{m=1}^{N} A_mQ, \quad (AQ)^0 = Q,$$
$$(AQ)^k = A((AQ)^{k-1}).$$

The sequence of sets is monotonically decreasing and by [14, Theorem 8.3]

$$\Gamma = (AQ)^\infty = \bigcap_{k\in\mathbb{N}} (AQ)^k = \lim_{k\to\infty} (AQ)^k$$

is the uniquely determined fractal generated by the contractions $(A_m)_{m=1}^N$. But on the other hand we assume that $\mu(A_mQ) = r^d$, where $m = 1, \ldots, N$, and from here we get that $d = \frac{\log N}{|\log r|}$. If $0 < d < 2$ then it follows by elementary geometrical reasoning that one can finds (sufficiently small) numbers $r > 0$ and suitably chosen natural numbers $N \in \mathbb{N}$ with the desired properties. □

Remark 4.3.3 *Our definition for the anisotropic $d-$set is a generalization of Farkas definition [18, Sect. 3.1]. In the following we recall his definition. If $j \in \mathbb{N}_0$ and $N_j \in \mathbb{N}_0$ we deal with sets of open rectangles $\{R_{jl} : l = 1, \ldots, N_j\}$ in $\mathbb{R}^n$ having sides parallel to the axes, the side length of the rectangle R_{jl} with respect to the x_i-axis is denoted by $r_i^{j,l}$ where $i = 1, \ldots, n$. We will always assume that the side lengths of the rectangles R_{jl} are ordered in the same way, for example $r_1^{j,l} \leq \ldots \leq r_n^{j,l}$ for any $j \in \mathbb{N}_0$ and any $l = 1, \ldots, N_j$.*
Let Q be a cube in $\mathbb{R}^n$ with side length 1, let $0 < d < n$, let $a = (a_1, \ldots, a_n)$ a given anisotropy and let $c_1, c_2 > 0$ given numbers.
Let $N_0 = 1$ and for any $j \in \mathbb{N}$ let N_j be a natural number satisfying

$$c_1 2^{jd} \leq N_j \leq c_2 2^{jd}.$$

A compact set $\Gamma \subset \mathbb{R}^n$ *is called a* **regular anisotropic** $\mathbf{d}-$**set** *(with respect to the anisotropy a) if for any* $j \in \mathbb{N}_0$ *there exists a finite sequence of open rectangles* $\{R_{jl} : l = 1, \dots, N_j\}$ *having sides parallel to the axes,* $R_{01} = \mathring{Q}$*, the interior of* Q*, such that:*

(i) there exists a constant $0 < c_0 \leq 1$ *such that for all* $i = 1, \dots, n$*, all* $j \in \mathbb{N}_0$ *and all* $l = 1, \dots, N_j$

$$(c_0 2^{-j})^{a_i} \leq r_i^{j,l} \leq 2^{-ja_i} \tag{4.31}$$

(ii) if $l \neq l'$ *then* $R_{jl} \cap R_{jl'} = \emptyset$

(iii) for any rectangle $R_{j+1,k}$ *there exists a rectangle* R_{jl}*,* $l = l(k)$*, such that* $R_{j+1,k} \subset R_{jl}$

(iv) for any $j \in \mathbb{N}_0$ *and any* $l = 1, \dots, N_j$

$$(vol\, R_{jl})^{\frac{d}{n}} = \sum_{R_{j+1,k} \subset R_{jl}} (vol\, R_{j+1,k})^{\frac{d}{n}} \tag{4.32}$$

(v)

$$\Gamma = \bigcap_{j=0}^{\infty} \bigcup_{l=1}^{N_j} \overline{R_{jl}}.$$

Let $n = 2$*, let* $a = (a_1, a_2)$ *a* $2-$*dimensional anisotropy and let* $0 < d < 2$*, then* Γ *is also an anisotropic* $d-$*set in the sense of Triebel [49, Sect. 5.2].*

Let $0 < d < n$ *and let* Γ *be the regular anisotropic* $d-$*set (with respect to the given anisotropy* $a = (a_1, \dots, a_n)$*) introduced above. Then there exists a Radon measure* μ *in* $\mathbb{R}^n$ *uniquely determined with* $\operatorname{supp} \mu = \Gamma$ *and*

$$\mu(\Gamma \cap R_{jl}) = (vol\, R_{jl})^{\frac{d}{n}}, \quad j \in \mathbb{N}_0 \quad \textit{and} \quad l = 1, \dots, N_j, \tag{4.33}$$

see [18, Theorem 3.5]. Let $B^a(x, 2^{-j}) = \{y \in \mathbb{R}^n : |y - x|_a \leq 2^{-j}\}$ *be an anisotropic ball like in Definition 4.3.1 with* $r = 2^{-j}$*. It is easy to see that* $B^a(x, 2^{-j}) \subset \{y \in \mathbb{R}^n : |y_i - x_i| \leq c 2^{-ja_i},\ i = 1, \dots, n\}$*. By* (4.31) $B^a(x, 2^{-j})$ *has a nonempty intersection with at most* N *rectangles* R_{jl}*,* $(l = 1, \dots, N_j)$*, where* N *is independent of* j *so that using* (4.33) *we get*

$$\mu(B^a(x, 2^{-j}) \cap \Gamma) \leq c' 2^{-jd}$$

where $c' > 0$ *is independent of* j*.*

If $0 < \kappa < 1$ then κR_{jl} denotes the rectangle concentric with R_{jl} and with side lengths respectively $\kappa r_1^{j,l}, \dots, \kappa r_n^{j,l}$. The regular anisotropic $d-$set introduced above equipped with the measure μ according to (4.33) *is called* **proper** *if there exist two numbers $0 < \kappa < 1$ and $0 < c \le 1$ such that*

$$\mu(\Gamma \cap \kappa R_{jl}) \geq c(vol\, R_{jl})^{\frac{d}{n}}, \quad j \in \mathbb{N}_0, \quad l = 1, \dots, N_j. \qquad (4.34)$$

Following the lines of the proof of [49, Sect. 5.13] it turns out that if Γ is generated by linear contractions and if $\Gamma \cap \mathring{Q} \neq \emptyset$ then Γ is proper. The Definition 4.3.1 covers also the feature of Γ to be proper, because by (4.30) *we have that*

$$\mu(B^a(x, 2^{-j}) \cap \Gamma) \geq c' 2^{-jd},$$

where $c' > 0$. So it is easy to see that Farkas regular anisotropic d-set with proper property is also an anisotropic d-set according to Definition 4.3.1.

Example 4.3.4 *Let $Q = [0,1]^2$ and let* log *be taken with respect to the base* 2*, let $1 < K_1 < K_2$ be natural numbers so that $\frac{K_1}{K_2} = 2k+1$ for some $k \in \mathbb{N}$, and let*

$$a_1 = \frac{2\log K_1}{\log(K_1K_2)}, \quad a_2 = \frac{2\log K_2}{\log(K_1K_2)}, \quad \kappa = \frac{1}{2}\log(K_1K_2).$$

We can see that $a_1, a_2 > 0$ and $a_1 + a_2 = 2$. Let $(A_m)_{m=1}^N$ be $N \geq 2$ contractions of $\mathbb{R}^2$ into itself specified by

$$A_m : x = (x_1, x_2) \longmapsto (\eta_1^m 2^{-\kappa a_1} x_1, \eta_2^m 2^{-\kappa a_2} x_2) + x_m \qquad (4.35)$$

for every $m = 1, \dots, N$ where η_2^m-is always 1 *and we choose $\eta_1^m = 1$ in the first K_2 columns, $\eta_1^m = -1$ in the second K_2 columns, then again $\eta_1^m = 1$ in the third K_2 columns and so on, and x_m in* (4.35) *is chosen such that we have the situation as depicted in Fig.2. We assume $A_m Q \subset Q$ for all $m = 1, \dots, N$, $A_m \mathring{Q} \cap A_{m'} \mathring{Q} = \emptyset$ if $m \neq m'$, and we suppose that the rectangles $A_m Q$ are located in the columns as indicated in Fig.2. Let*

$$AQ = (AQ)^1 = \bigcup_{m=1}^{N} A_m Q, \quad (AQ)^0 = Q,$$

$$(AQ)^k = A((AQ)^{k-1}).$$

The sequence of sets is monotonically decreasing and by [49, Theorem 4.2]

$$\Gamma = (AQ)^\infty = \bigcap_{k \in \mathbb{N}} (AQ)^k = \lim_{k \to \infty} (AQ)^k$$

is the uniquely determined fractal generated by the contractions $(A_m)_{m=1}^N$*. Under these assumption the resulting* Γ *is the graph of a continuous function, see [49, Sect. 4.21]. Moreover,* Γ *is an isotropic d-set where* $d = dim_H \Gamma = 2 - \min(1, \frac{a_2}{a_1})$*, see [49, Sect. 4.22, 4.23], and in the Farkas' sense it is a regular anisotropic d-set with* $d = a_1$*. In our example in Fig.2 with* $N = 7$ *we calculated d in the same way like in the proof of Proposition 4.3.2 and we get that* $d = \frac{\log 7}{\kappa}$*.*

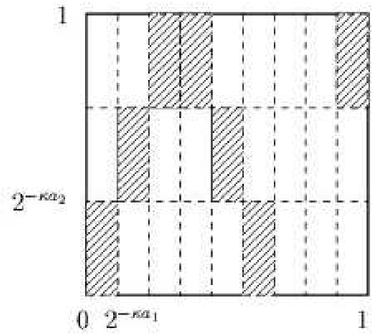


Fig.2

Example 4.3.5 *Let* $A_1,\ A_2$ *be the affine contractions on* $\mathbb{R}^n$ *which map the unit square onto the rectangles* $R_1,\ R_2$ *of sides* 2^{-a_1} *and* 2^{-a_2} *where* $0 < a_2 < a_1$ *and* $a_1 + a_2 = 2$ *as in Figure 3 with the distance* $\lambda \geq 0$*. In the same way as in Example 4.3.4 we have that* $d = 1$*, in the sense of our definition. In [18, Sect. 3.1] we can see that Farkas also has for this example* $d = 1$*.*

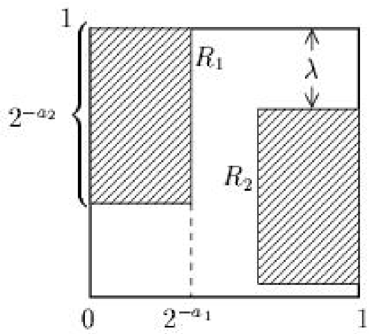


Fig.3

4.4 Main assertion

We are now prepared to formulate our main result.

Theorem 4.4.1 *Let the anisotropic d-set* Γ *and* μ *be given according to* (4.30)*, and*

$$0 < d < n, \quad 1 < p < \infty, \quad \frac{n}{p} \geq s > \frac{n-d}{p}.$$

Let $a_k = a_k(tr_\Gamma)$ be the approximation numbers of the compact operator tr_Γ according to (4.12). *Then there exist numbers $c, c' > 0$ so that for all $k \in \mathbb{N}$*

$$ck^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}} \leq a_k(tr_\Gamma : B^{s,a}_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma)) \leq c'k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}. \tag{4.36}$$

Proof. Step 1 First we prove the right-hand side of the estimate (4.36) in Theorem 4.4.1. Again we use the wavelet expansion (3.29), (3.30). For fixed $\beta \in \mathbb{N}^n_0$ we put

$$tr^\beta_\Gamma f = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n}^{\Gamma} \lambda^\beta_{jm}(f) k^\beta_{jm} \tag{4.37}$$

and

$$tr^{\beta,J}_\Gamma f = \sum_{j\leq J} \sum_{m\in\mathbb{Z}^n}^{\Gamma} \lambda^\beta_{jm}(f) k^\beta_{jm}, \tag{4.38}$$

where the second sum has the same meaning as the last sum in (4.18). By the same reasoning as in (4.27) and (4.29) but now for fixed β we have for $f \in B^{s,a}_p(\mathbb{R}^n)$ with norm of at most 1,

$$\|(tr^\beta_\Gamma - tr^{\beta,J}_\Gamma) f | L_p(\Gamma)\| \leq c 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} \mu_J^{\frac{1}{p}}. \tag{4.39}$$

By Definition 4.3.1 there exists a constant $c > 0$ independent of $j \in \mathbb{N}_0$ with $\mu(Q^a_{jm} \cap \Gamma) \leq c 2^{-jd}$ and we obtain that

$$\|(tr^\beta_\Gamma - tr^{\beta,J}_\Gamma) f | L_p(\Gamma)\| \leq c 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} 2^{-J\frac{d}{p}}. \tag{4.40}$$

In definition (4.21) put $L = tr^{\beta,J}_\Gamma$, $T = tr^\beta_\Gamma$, and note that for $j \in \mathbb{N}_0$,

$$\operatorname{rank}\left(\sum_{m\in\mathbb{Z}^n}^{\Gamma} \lambda^\beta_{jm}(f) k^\beta_{jm} \right) \leq c 2^{jd}. \tag{4.41}$$

Thus we obtain by (4.38) that

$$\operatorname{rank}(tr^{\beta,J}_\Gamma) \leq c \sum_{j\leq J} 2^{jd} \leq c' 2^{Jd}. \tag{4.42}$$

Then (4.41) implies that there are two positive numbers c and c' such that

$$a_{c2^{Jd}}(tr^\beta_\Gamma) \leq c' 2^{-\delta a \beta} 2^{J(\frac{n}{p}-s)} 2^{-J\frac{d}{p}}. \tag{4.43}$$

For $k \in \mathbb{N}$ there are numbers $J_k \in \mathbb{N}$ such that

$$2^{J_k d} \sim k \quad \text{with} \quad J_1 \leq J_2 \leq \cdots \leq J_n \leq \cdots; \tag{4.44}$$

inserted in (4.43) this leads to

$$a_{ck}(tr_\Gamma^\beta) \le c2^{-\delta a\beta} 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}}. \tag{4.45}$$

Let $\varepsilon > 0$, for given $k \in \mathbb{N}$ we apply (4.45) to $k_\beta \in \mathbb{N}$ with $k_\beta \sim 2^{-\varepsilon a\beta} k$. Then it follows by the additivity property of approximation numbers and from (4.45) that

$$\begin{aligned} a_{ck}(tr_\Gamma^\beta) &\le \sum_{\beta\in\mathbb{N}_0^n} a_{k_\beta}(tr_\Gamma^\beta) \\ &\le c' \sum_{\beta\in\mathbb{N}_0^n} 2^{-\delta a\beta} 2^{J_{k_\beta}(\frac{n}{p}-s)} (2^{-\varepsilon a\beta} k)^{-\frac{1}{p}} \\ &\le c'' 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}} \sum_{\beta\in\mathbb{N}_0^n} 2^{-a\beta(\delta-\frac{\varepsilon}{p})} \\ &\le c''' 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}} \end{aligned} \tag{4.46}$$

for $\varepsilon > 0$ small. We used $s \le \frac{n}{p}$, such that $J_{k_\beta}(\frac{n}{p} - s) \le J_k(\frac{n}{p} - s)$. Finally (4.44) implies

$$a_{ck}(tr_\Gamma^\beta) \le c''' k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}} \tag{4.47}$$

and so we finished the proof of the right-hand side of the estimate (4.36).

Step 2 To verify the left-hand side of the estimate (4.36) we closely follow the argument in [54, Sect. 4.4] for the isotropic case. Let $J \in \mathbb{N}$ and $c > 0$ be suitably chosen numbers such that there are lattice points

$$\gamma_{j,l} = 2^{(-j-J)a} m \quad \text{with} \quad m \in \mathbb{Z}^n, \quad l = 1, \dots, M_j \quad \text{where} \quad M_j \sim 2^{jd} \tag{4.48}$$

with

$$\operatorname{dist}(\gamma_{j,l}, \Gamma) \le c2^{-j} \quad \text{and disjoint anisotropic balls} \quad B^a(\gamma_{j,l}, c2^{-j+1}). \tag{4.49}$$

With k as in (3.11) we put for $j \in \mathbb{N}_0$,

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} k(2^{ja}(x - \gamma_{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n. \tag{4.50}$$

Then we obtain by Theorem 2.3.1

$$\|f_j^a | B_p^{s,a}(\mathbb{R}^n)\| \sim 2^{j(s-\frac{n}{p})} \left(\sum_{l=1}^{M_j} 2^{-j(s-\frac{n}{p})} |c_{jl}|^p \right)^{\frac{1}{p}} = \left(\sum_{l=1}^{M_j} |c_{jl}|^p \right)^{\frac{1}{p}} \tag{4.51}$$

and

$$\begin{aligned} \|f_j^a|L_p(\Gamma)\| &= \left(\int_\Gamma |f_j^a(x)|^p \mu(dx)\right)^{1/p} \\ &\sim 2^{-j(s-\frac{n}{p})}\left(\sum_{l=1}^{M_j} |c_{jl}|^p \int_\Gamma k^p(2^{ja}(x-\gamma_{j,l}))\mu(dx)\right)^{1/p} \\ &\sim 2^{-j(s-\frac{n}{p})}\left(\sum_{l=1}^{M_j} |c_{jl}|^p \int_{\Gamma\cap B^a(\gamma_{j,l},c2^{-j})} k^p(2^{ja}(x-\gamma_{j,l}))\mu(dx)\right)^{1/p} \\ &\geq c2^{-j(s-\frac{n}{p})}2^{-j\frac{d}{p}}\left(\sum_{l=1}^{M_j} |c_{jl}|^p\right)^{1/p} \end{aligned} \tag{4.52}$$

using our assumption (4.30) in the last estimate. Hence

$$\|f_j^a|L_p(\Gamma)\| \geq c2^{-j(s-\frac{n}{p})}2^{-\frac{jd}{p}} \quad \text{if} \quad \|f_j^a|B_p^{s,a}(\mathbb{R}^n)\| \sim 1. \tag{4.53}$$

Now let T be an arbitrary linear operator,

$$T: \quad B_p^{s,a}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma) \quad \text{with} \quad \text{rank}\, T \leq M_j - 1. \tag{4.54}$$

Then we can find a function f_j^a according to (4.50) with norm 1 in $B_p^{s,a}(\mathbb{R}^n)$ and $Tf_j^a = 0$. Consequently, by (4.52) and (4.53),

$$\begin{aligned} \|tr_\Gamma - T\| &= \sup\left\{\|(tr_\Gamma - T)f \mid L_p(\Gamma)\| : \|f|B_p^{s,a}(\mathbb{R}^n)\| \sim 1\right\} \\ &\geq \|(tr_\Gamma - T)f_j^a|L_p(\Gamma)\| \\ &= \|f_j^a|L_p(\Gamma)\| \\ &\geq c2^{-j(s-\frac{n}{p})-j\frac{d}{p}}. \end{aligned} \tag{4.55}$$

As this is true for all T according to (4.54), we obtain

$$\begin{aligned} a_{M_j}(tr_\Gamma) &= \inf\{\|tr_\Gamma - T\| \; : \text{rank}\, T \leq M_{j-1}\} \\ &\geq c2^{-j(s-\frac{n}{p})-j\frac{d}{p}}. \end{aligned} \tag{4.56}$$

For $k \in \mathbb{N}$ there are numbers $j_k \in \mathbb{N}$ such that

$$2^{j_k d} \sim k \quad \text{with} \quad j_{k_1} \leq j_{k_2} \leq \cdots j_{k_n} \leq \cdots,$$

inserted in (4.53) we obtain

$$\begin{aligned} a_k(tr_\Gamma) &\geq c2^{j_k(s-\frac{n}{p})}k^{-\frac{1}{p}} \\ &\geq c'k^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}, \end{aligned} \tag{4.57}$$

i.e. the left-hand side of the estimate (4.36). □

Remark 4.4.2 *Let* Γ *be the anisotropic d-set considered in [18, Sect. 3.1], see Remark 4.3.3. Farkas proved in this situation that*

$$e_k(tr_\Gamma : B^{\delta+\frac{n-d}{p_1},a}_{p_1 q}(\mathbb{R}^n) \longrightarrow L_{p_2}(\Gamma)) \sim ck^{-\frac{\delta}{d}}$$

where $0 < p_1, p_2, q \le \infty$, *and* $\delta > 0$. *Let* $p_1 = p_2 = q = p$ *and* $s = \delta + \frac{n-d}{p}$, *then*

$$e_k(tr_\Gamma : B^{s,a}_p(\mathbb{R}^n) \longrightarrow L_p(\Gamma)) \sim ck^{\frac{1}{d}(\frac{n}{p}-s)-\frac{1}{p}}.$$

So we have the same results for the entropy and approximation numbers in the special case $p_1 = p_2$ *which is not surprising, see* (4.24), *but cannot be expected for* $p_1 \neq p_2$.
In view of the isotropic result [54, Theorem 2, Remark 9], if we restrict the outcome [54] to the classical example of a compact d-set with $0 < d < n$, *then we have the same result like in the anisotropic setting.*

5 Eigenvalue distribution of semi-elliptic operators

5.1 Introduction

Let us consider a differential expression with real coefficients $A(D) = \sum a_\alpha D^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, and $|\alpha| = \sum_{i=1}^n \alpha_i$. Let $l = (l_1, \ldots, l_n)$, $(l_k > 0, 1 \leq k \leq n)$ be a fixed multi-index. We write $(\alpha : 2l) = \sum_{k=1}^n \frac{\alpha_k}{2l_k}$. We introduce the following differential operator:

$$A(D)u = \sum_{(\alpha:2l)=1} a_\alpha D^\alpha u.$$

$A(D)$ is said to be semi-elliptic if the corresponding polynomial

$$A(\xi) = \sum_{(\alpha:2l)=1} a_\alpha \xi^\alpha > 0$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$.

First we give a physical background of the study of such operators. Let Ω be a bounded domain in the plane $\mathbb{R}^2$ with C^∞ boundary $\partial\Omega$, interpreted as a membrane fixed at its boundary. Vibrations of such a membrane in $\mathbb{R}^3$ are measured by the defection $v(x,t)$, where $x = (x_1, x_2) \in \Omega$, and $t \geq 0$ stands for the time. In other words, the point $(x_1, x_2, 0)$ in $\mathbb{R}^3$ with $(x_1, x_2) \in \Omega$ of the membrane at rest, is defected to $(x_1, x_2, v(x,t))$. Up to constants the usual physical description is given by

$$\Delta v(x,t) = m(x) \frac{\partial^2 v(x,t)}{\partial t^2}, \quad x \in \Omega, \quad t \geq 0, \tag{5.1}$$

and

$$v(y,t) = 0 \quad \text{if} \quad y \in \partial\Omega, \quad t \geq 0, \tag{5.2}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and the right-hand side of (5.1) is Newton's law with the mass density $m(x)$. To find the eigenfrequencies one has to insert $v(x,t) = u(x)e^{i\lambda t}$ with $\lambda \in \mathbb{R}$ in (5.1) and obtains

$$-\Delta u(x) = \lambda^2 m(x) u(x), \quad x \in \Omega; \quad u(y) = 0 \quad \text{if} \quad y \in \partial\Omega, \tag{5.3}$$

where one is interested in non-trivial solutions $u(x)$. Hence one asks for eigenfunctions and eigenvalues of the operator

$$B = (-\Delta)^{-1} \circ m(\cdot), \tag{5.4}$$

where $(-\Delta)^{-1}$ is the inverse of the *Dirichlet Laplacian* $-\Delta$. We use the notation *Dirichlet Laplacian* always with the understanding that vanishing boundary data at $\partial\Omega$ are incorporated into domains of definition for $-\Delta$ in the function spaces considered, preferably $B^s_{pq}(\Omega)$ and $H^s_p(\Omega)$ with $1 < p \leq \infty$ and $s > \frac{1}{p}$ (this will be specified in greater detail in the next subsection). If ϱ is a positive eigenvalue of B then $\lambda = \varrho^{-\frac{1}{2}}$ is the related eigenfrequency. We are interested in the problem of what happens when the mass density $m(x)$ shrinks to a fractal set Γ and a related Radon measure μ with

$$supp\ \mu = \Gamma \subset \Omega. \tag{5.5}$$

Hence we ask for eigenfrequencies and eigenfunctions of drums with a fractal membrane. This is what we call fractal drums and fractal Laplacians (extending this notation to $n \in \mathbb{N}$, where Ω let be a bounded domain in $\mathbb{R}^n$).
We want to mention that the notation of fractal drums has several meanings. As for the study of fractal membranes in smooth domains, we know only a few further papers in literature, see T. Fujita [22], K. Naimark and M. Solomyak [33] and [34], M. Solomyak and E. Verbitsky [40], and the more recent article of D.E. Edmunds and H. Triebel [12].
Further results on the vibration of "fractal drums" are obtained in different settings. Maybe the best known version is connected with the study of the Laplacian on a fractal, as it is done for example in the works of J. Kigami and M.L. Lapidus, see [27], [30]. A detailed discussion on these different aspects concerning fractal drums can de found in [49, Sect. 26.2, 30.1-30.5].
Our motivation in section 5.2 is the consideration in [54]. H. Triebel proved in [54] for the fractal elliptic operator of type

$$B_s = (-\Delta + id)^{-s} \circ id^\Gamma, \tag{5.6}$$

that B_s is a compact, non-negative, self-adjoint operator in $W^s_2(\mathbb{R}^n)$, where

$$id^\Gamma = id_\Gamma \circ tr_\Gamma,$$

and $tr_\Gamma : W^s_2 \to L_2(\Gamma)$ is the trace operator, and id_Γ is the dual of the trace operator. If we restrict the outcome to the classical example of a compact $d-$set with $0 < d < n$ and $n - d < 2s \leq n$, we get that

$$\lambda_k(B_s) \sim k^{-\frac{1}{d}(d+2s-n)}, \tag{5.7}$$

see [54, Theorem 3, Remark 10].
An important step in anisotropic function spaces was made by Farkas in the papers [18] and [15]. He studied the operator

$$A^{-1} \circ tr^{\Gamma} \tag{5.8}$$

where

$$tr^{\Gamma} : \; B^{\frac{2-d}{p},a}_{p1}(\mathbb{R}^2) \to B^{-\frac{2-d}{p'},a}_{p\infty}(\mathbb{R}^2), \tag{5.9}$$

and A^{-1} is the inverse of

$$Au(x) = (-1)^{t_1} \frac{\partial^{2t_1} u(x)}{\partial x_1^{2t_1}} + (-1)^{t_2} \frac{\partial^{2t_2} u(x)}{\partial x_n^{2t_2}} + u(x), \tag{5.10}$$

and proved that the operator $A^{-1} \circ \; tr^{\Gamma}$ is compact, non-negative, and self-adjoint in $W^{t,a}_2(\mathbb{R}^2)$ and its positive eigenvalues can be estimated by

$$\lambda_k(A^{-1} \circ tr^{\Gamma}) \sim ck^{-\frac{1}{d}(d+2t-2)}.$$

Our main aim in the following section is to study operators of type (5.8), in the case $\mathbb{R}^n$, where we follow the ideas in [54].

5.2 Main assertion

Let Γ be an anisotropic $d-$set with respect to the anisotropy $a = (a_1, \dots, a_n)$, then by (4.3), with $p = 2$, and (1.14)

$$tr_{\Gamma} : \; H^{s,a}_2(\mathbb{R}^n) \hookrightarrow L_2(\Gamma). \tag{5.11}$$

By (1.14), (4.7) and (4.8) we have that

$$id_{\Gamma} : \; L_2(\Gamma) \hookrightarrow H^{-s,a}_2(\mathbb{R}^n). \tag{5.12}$$

As a consequence,

$$tr^{\Gamma} = id_{\Gamma} \circ tr_{\Gamma} : \; H^{s,a}_2(\mathbb{R}^n) \hookrightarrow H^{-s,a}_2(\mathbb{R}^n). \tag{5.13}$$

Let $s_1, \dots, s_n \; \in \mathbb{N}$ and let $s \in \mathbb{R}$ be defined by

$$\frac{1}{s} = \frac{1}{n}\left(\frac{1}{s_1} + \cdots + \frac{1}{s_n}\right). \tag{5.14}$$

Let A be the operator defined by

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x) \tag{5.15}$$

where $x \in \mathbb{R}^n$. Using elementary properties of the Fourier transform we have

$$Au = \left((1 + \xi_1^{2s_1} + \cdots + \xi_n^{2s_n}) \hat{u} \right)^{\vee}$$

for any $u \in S'(\mathbb{R}^n)$.
It is well known, see for example [31], that A is a lift operator for the scale $B^{t,a}_{pq}(\mathbb{R}^n)$, $t \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. More precisely A maps any space $B^{t,a}_{pq}(\mathbb{R}^n)$ onto $B^{t-2s,a}_{pq}(\mathbb{R}^n)$ and $\|A(\cdot)|B^{t-2s,a}_{pq}(\mathbb{R}^n)\|$ is an equivalent quasi-norm on $B^{t-2s,a}_{pq}(\mathbb{R}^n)$, the inverse A^{-1} of A has to be understood in this way.

Theorem 5.2.1 *Let $\Gamma \subset \mathbb{R}^n$ be an anisotropic d-set according to Definition 4.3.1 with respect to the anisotropy a. Let tr^Γ be the operator given by* (5.13), $s_i \in \mathbb{N}$, $i = 1, \ldots, n$, $\frac{1}{s} = \frac{1}{n}\left(\frac{1}{s_1} + \cdots + \frac{1}{s_1}\right)$, *A given by*

$$Au(x) = (-1)^{s_1} \frac{\partial^{2s_1} u(x)}{\partial x_1^{2s_1}} + \cdots + (-1)^{s_n} \frac{\partial^{2s_n} u(x)}{\partial x_n^{2s_n}} + u(x)$$

and

$$0 < d < n, \ \frac{n}{2} \ge s > \frac{n-d}{2}. \tag{5.16}$$

Then

$$T = A^{-1} \circ tr^\Gamma \tag{5.17}$$

is a compact, non-negative self-adjoint operator in $W_2^{s,a}(\mathbb{R}^n)$ and with null space

$$N(T) = \{ f \in W_2^{s,a}(\mathbb{R}^n) : tr_\Gamma f = 0 \}. \tag{5.18}$$

Let $(\lambda_k)_{k \in \mathbb{N}}$ be the sequence of all positive eigenvalues of T, repeated according to multiplicity and ordered by their magnitude. Then

$$\lambda_k \sim k^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}. \tag{5.19}$$

We begin the proof of Theorem 5.2.1 with some preparation.

Lemma 5.2.2 *Let s be given by* (5.14) *and A the operator from* (5.15).

1. *There exists a constant $c > 0$ such that $(Au, u)_{L_2(\mathbb{R}^n)} \ge c\|u|L_2(\mathbb{R}^n)\|$ for any $u \in L_2(\mathbb{R}^n)$.*

2. *There exist two constants $c_1, c_2 > 0$ such that*

$$c_1 \|u|W_2^{s,a}(\mathbb{R}^n)\|^2 \le (Au, u)_{L_2(\mathbb{R}^n)} \le c_2 \|u|W_2^{s,a}(\mathbb{R}^n)\|^2.$$

P r o o f We closely follow the ideas in [18, Sect. 6.2, Lemma 6.3]. Farkas proved the lemma for the case if $n = 2$, and now we extend it to the case $n \in \mathbb{N}$.
Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and

$$(A\varphi, \varphi)_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (A\varphi)(x)\overline{\varphi(x)}dx.$$

After integration by parts we have

$$(A\varphi, \varphi)_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\left| \frac{\partial^{s_1}\varphi(x)}{\partial x_1^{s_1}} \right|^2 + \cdots + \left| \frac{\partial^{s_n}\varphi(x)}{\partial x_n^{s_n}} \right|^2 + |\varphi(x)|^2 \right) dx$$

and the conclusion of the lemma follows immediately using the density of $C_0^\infty(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ and in $W_2^{s,a}(\mathbb{R}^n)$, and the definition of the space $W_2^{s,a}(\mathbb{R}^n)$, see (1.13). □

We finally can prove Theorem 5.2.1.
P r o o f (of Theorem 5.2.1)
Step 1. In this step we prove that T given by (5.17) is compact, non-negative self-adjoint operator in $W_2^{s,a}(\mathbb{R}^n)$. By (5.17) we have that $T = A^{-1} \circ tr^\Gamma$, where $tr^\Gamma = id_\Gamma \circ tr_\Gamma$,

$$\begin{aligned} tr_\Gamma &: W_2^{s,a}(\mathbb{R}^n) \hookrightarrow L_2(\Gamma) \\ id_\Gamma &: L_2(\Gamma) \hookrightarrow H_2^{-s,a}(\mathbb{R}^n) \\ A^{-1} &: H_2^{-s,a}(\mathbb{R}^n) \hookrightarrow W_2^{s,a}(\mathbb{R}^n). \end{aligned} \tag{5.20}$$

By Lemma 5.2.2 we have that the operator A is positive definite as an operator acting in $L_2(\mathbb{R}^n)$ and we may fix the norm in $W_2^{s,a}(\mathbb{R}^n)$ by $\|A^{1/2}(\cdot)|L_2(\mathbb{R}^n)\|$ and a corresponding scalar product. By Proposition 4.1.1, (1.14) and (4.2) there exists a constant $c > 0$ such that

$$\|tr_\Gamma \varphi | L_2(\Gamma)\| \le c\|\varphi|W_2^{s,a}(\mathbb{R}^n)\| \quad \text{for all} \quad \varphi \in W_2^{s,a}(\mathbb{R}^n).$$

Defining

$$q(f,g) = \int_\Gamma f(\gamma)\overline{g(\gamma)}d\mu(\gamma) \quad \text{for any} \quad f, g \in W_2^{s,a}(\mathbb{R}^n),$$

it is clear that $q(\cdot,\cdot)$ is a non-negative quadratic form in $W_2^{s,a}(\mathbb{R}^n)$. Then there exists a non-negative and self-adjoint operator $\tilde{T}$ uniquely determined such that

$$q(f,g) = (\tilde{T}f, g)_{W_2^{s,a}(\mathbb{R}^n)} \quad \text{for any} \quad f, g \in W_2^{s,a}(\mathbb{R}^n),$$

see for example [48, p.91]. Furthermore,

$$\|tr_\Gamma f|L_2(\Gamma)\| = \|\sqrt{\tilde{T}}f|W_2^{s,a}(\mathbb{R}^n)\| \tag{5.21}$$

where $\sqrt{\tilde{T}} = \tilde{T}^{1/2}$. So it remains to prove that the above operator is the same as in (5.17). Let $f \in W_2^{s,a}(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$. Then by (4.4), (4.8) and (5.13)

$$\begin{aligned}(tr^\Gamma f, \varphi)_{L_2(\mathbb{R}^n)} = \int_\Gamma f(\gamma)\overline{\varphi(\gamma)}d\mu(\gamma) &= (\tilde{T}f, \varphi)_{W_2^{s,a}(\mathbb{R}^n)} \\ &= (A^{1/2}\tilde{T}f, A^{1/2}\varphi)_{L_2(\mathbb{R}^n)} \\ &= (A\tilde{T}f, \varphi)_{L_2(\mathbb{R}^n)} \end{aligned} \tag{5.22}$$

the second equality in (5.22) being justified by the fact that we fixed the scalar product in $W_2^{s,a}(\mathbb{R}^n)$ by

$$(u, v)_{W_2^{s,a}(\mathbb{R}^n)} = (A^{1/2}u, A^{1/2}v)_{L_2(\mathbb{R}^n)}.$$

Considered as a dual pairing in $(D(\mathbb{R}^n), D'(\mathbb{R}^n))$ we obtain $A\tilde{T}f = tr^\Gamma f$, and we have that $\tilde{T} = T$ by (5.17).
The compactness is a consequence of Theorem 4.4.1 and (5.20).
Step 2. We prove that there is a number $c > 0$ such that

$$\lambda_k \le ck^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}. \tag{5.23}$$

By (4.8) the identification operator id_Γ is the dual of the trace operator tr_Γ. By the usual assertion for dual operators, Proposition 4.2.2 (*i*), and Theorem 4.4.1 we have

$$a_k(id_\Gamma) = a_k(tr_\Gamma) \sim k^{\frac{1}{d}(\frac{n}{2}-s)-\frac{1}{2}}, \quad k \in \mathbb{N}, \tag{5.24}$$

and we need that $\frac{n}{2} \ge s > \frac{n-d}{2}$. By (5.20) and the multiplication property for approximation numbers, Lemma 4.2.1 (*iii*), one obtains

$$a_{2k}(T) \le c\, a_k(tr_\Gamma)a_k(id_\Gamma) \sim k^{-\frac{1}{d}(2s-n+d)}. \tag{5.25}$$

By Proposition 4.2.2 (*ii*) and Step 2 we have that the approximation numbers of T coincide with its eigenvalues. Then (5.23) follows from (5.25).
Step 3. To obtain the converse of (5.23) we use the same argument as in Theorem 4.4.1, Step 2, now with $p = 2$. Based on (4.48)-(4.49) we put

$$f_j^a(x) = \sum_{l=1}^{M_j} c_{jl}2^{-j(s-\frac{n}{2})}k(2^{ja}(x-\gamma_{j,l})), \quad c_{jl} \in \mathbb{C},\ x \in \mathbb{R}^n, \tag{5.26}$$

where $M_j \sim 2^{jd}$. By (5.21) and by (4.51)-(4.53)

$$\|\sqrt{T} f_j^a | W_2^{s,a}(\mathbb{R}^n)\| \sim c 2^{-j(s-\frac{n}{2})} 2^{-j\frac{d}{2}} \quad \text{if } \|f_j^a | W_2^{s,a}(\mathbb{R}^n)\| \sim 1. \tag{5.27}$$

By the same arguments as in connection with (4.54)-(4.55) we obtain

$$a_{M_j}(\sqrt{T}) \geq c 2^{-j(s-\frac{n}{2})} 2^{-j\frac{d}{2}}. \tag{5.28}$$

By (4.57) and by $a_k(\sqrt{T}) = \lambda_k^{\frac{1}{2}}$ we obtain

$$\lambda_k \geq c k^{-\frac{1}{d}(2s-n+d)}, \quad k \in \mathbb{N}. \tag{5.29}$$

□

Remark 5.2.3 *(i) Let* Γ *be the anisotropic* $d-$*set considered in [18, Sect. 3.1], see Remark 4.3.3. Farkas proved in [18, Sect.4] for the operator* (5.8) *that*

$$\lambda_k(A^{-1} \circ tr^\Gamma) \sim c k^{-\frac{1}{d}(d+2t-2)}.$$

If we take the case $n = 2$ *and* $t = s$ *we have the same result*

$$\lambda_k \sim k^{-\frac{1}{d}(2t-2+d)}.$$

(ii) If we restrict to the case $n = 2$, $s_1 = 1$ *and* $s_2 = 2$ *then*

$$Au(x) = -\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^4 u(x)}{\partial x_2^4} + u(x), \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{5.30}$$

Let again Γ *be the anisotropic* $d-$*set considered in [18, Sect. 3.1], with respect to the anisotropy* $a = (\frac{4}{3}, \frac{2}{3})$. *Farkas obtained in [16, Sect. 4] that:*

$$\lambda_k(A^{-1} \circ tr^\Gamma) \sim c k^{-\frac{1}{d}(d+\frac{2}{3})},$$

where tr^Γ *is given by* (5.9) *and* A^{-1} *is the inverse of* (5.30). *For this special case we have also the same result. Operators of this type have been investigated by H.Triebel in [46] and by V.Shevchik in [39].*

(iii) In view of the isotropic results [54, Theorem 3, Remark 10] for the operator

$$B_s = (id - \Delta)^{-s} \circ id^\mu$$

we have the same results like in the anisotropic case if we restrict the outcome to the classical example of a compact $d-$*set with* $0 < d < n$ *and* $n - d < 2s \leq n$, *see* (5.7).

Bibliography

[1] M.Z. Berkolaĭko and I.Ya. Novikov. Unconditional bases in spaces of functions of anisotropic smoothness. *Trudy Mat. Inst. Steklov*, 204:35–51, 1993. Russian; Engl. transl. : Proc. Steklov Inst. Math. 1994, no. 3 (204), 27–41.

[2] M.Z. Berkolaĭko and I.Ya. Novikov. Bases of wavelets, and linear operators in anisotropic Lizorkin-Triebel spaces. *Trudy Mat. Inst. Steklov*, 210:5–30, 1995. Russian.

[3] O.V. Besov, V.P. Il'in, and S.M. Nikol'skiĭ. *Integral representations of functions, and embedding theorems (Russian).* Nauka, Moscow, 1975.

[4] M. Bownik. Anisotropic Hardy spaces and wavelets. *Mem. Amer. Math. Soc.*, 164(781):122 p., 2003.

[5] M. Bownik. Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces. *Trans. Amer. Math. Soc.*, 358(4):1469–1510, 2006.

[6] M. Bownik. Atomic and molecular decompositions of anisotropic Besov spaces. *Math. Z.*, 250(3):539-571, 2005.

[7] S. Dachkovski. Anisotropic function spaces and related semi-linear hypoelliptic equations. *Math. Nachr.*, 248-249(1):40–61, 2003.

[8] B.Carl, I.Stephani. *Entropy, Compactness and the Approximation of Operators.* Cambridge University Press, 1990.

[9] P. Dintelmann. Classes of Fourier multipliers and Besov-Nikolskij spaces. *Math. Nachr.*, 173:115–130, 1995.

[10] P. Dintelmann. *On Fourier multipliers between anisotropic weighted function spaces.* PhD thesis, TH Darmstadt, Germany, 1995.

[11] D.E. Edmunds and H. Triebel. *Function Spaces, Entropy Numbers, Differential Operators.* Cambridge Univ. Press, Cambridge, 1996.

[12] D.E. Edmunds and H. Triebel. Spectral theory for isotropic fractal drums. *C.R. Acad. Sci. Paris 326*, 1269-1274, 1998.

[13] D.E. Edmunds and W.D. Evans. *Spectral theory and differential operators.* Oxford Univ. Press, Oxford, 1987.

[14] K.J. Falconer. *The geometry of fractal sets.* Cambridge University Press, Cambridge, 1985.

[15] W. Farkas and H. Triebel. The distribution of eigenfrequencies of anisotropic fractal drums. *Math. Soc. (2) 60*, 224-236, 1999.

[16] W. Farkas. The behaviour of the eigenvalues for a class of operators related to some self-affine fractals in $\mathbb{R}^2$. *Z.Anal.Anwendungen* ,18:875-893, 1999.

[17] W. Farkas. Atomic and subatomic decompositions in anisotropic function spaces. *Math. Nachr.*, 209:83–113, 2000.

[18] E.W. Farkas. *Eigenvalue distribution of some fractal semi-elliptic differential operators. Math. Z.*, 236:291–320, 2001.

[19] M. Frazier and B. Jawerth. Decomposition of Besov spaces. *Indiana Univ. Math. J.*, 34(4):777–799, 1985.

[20] M. Frazier and B. Jawerth. A discrete transform and decomposition of distribution spaces. *J. Funct. Anal.*, 93(1):34–170, 1990.

[21] M. Frazier, B. Jawerth, and G. Weiss. Littlewood-Paley theory and the study of function spaces. *Mem. Amer. Math. Soc.*, 79:132 p., 1991.

[22] T. Fujita, 'A fractal dimension, self similarity and generalized diffusion operators', in: Probabilistic methods in mathematical physics. *Academic Press, Boston*, 83:90 p., 1987.

[23] G. Garrigós, R. Hochmuth, and A. Tabacco. Wavelet characterizations for anisotropic Besov spaces with $0 < p < 1$. *Proc. Edinb. Math. Soc. (2)*, 47(3):573–595, 2004.

[24] G. Garrigós and A. Tabacco. Wavelet decompositions of anisotropic Besov spaces. *Math. Nachr.*, 239-240:80–102, 2002.

[25] R. Hochmuth. Wavelet characterizations for anisotropic Besov spaces. *Appl. Comput. Harmon. Anal.*, 12(2):179–208, 2002.

[26] J. Johnsen. Pointwise multiplication of Besov and Triebel-Lizorkin spaces. *Math. Nachr.*, 175:85–133, 1995.

[27] J. Kigami and M.L. Lapidus. Weyl's problem and the spectral distribution of Laplacians on p.c.f.self-similar fractals. *Comm.Math.Phys. 158*, 93:125, 1993.

[28] G. Kyriazis. Multilevel characterizations of anisotropic function spaces. *SIAM J. Math. Anal.*, 36(2):441–462, 2004.

[29] G. Kyriazis and P. Petrushev. On the construction of frames for Triebel-Lizorkin and Besov spaces. *Proc. Amer. Math. Soc.*, 134(6):1759–1770, 2006.

[30] M.L. Lapidus. Vibrations of fractal drums. The Riemann hypothesis, waves in fractal media, and the Weyl-Berry conjecture, in:Ordinary and partial differential equations. *Pitman Research Notes in Math. 289, London, Longman Sci. Techn.*, 126:209, 1993.

[31] H.-G. Leopold. Boundedness of anisotropic pseudodifferential operators in function spaces of Besov-Hardy-Sobolev type. *Z. Anal. Anwendungen*, 5:409–417, 1986.

[32] Y. Meyer. *Wavelets and operators*, volume 37 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1992.

[33] K. Naimark and M. Solomyak. On the eigenvalue behaviour for a class of operators related to self-similar measures on $\mathbb{R}^d$, *C.R.Acad.Sci, Paris 319*, 837-842, 1994.

[34] K. Naimark and M. Solomyak. The eigenvalue behaviour for the boundary value problems related to self-similar measures on $\mathbb{R}^d$, *Math. Research Letters 2*, 279-298, 1995.

[35] S.M. Nikol'skiĭ. *Approximation of functions of several variables and embedding theorems (Russian).* Nauka, Moscow, 1977.

[36] Th. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* De Gruyter, Berlin, 1996.

[37] H.-J. Schmeißer and H. Triebel. *Topics in Fourier Analysis and Function Spaces.* Wiley, Chichester, 1987.

[38] A. Seeger. A note on Triebel-Lizorkin spaces. In *Approximation and function spaces (Warsaw, 1986)*, volume 22 of *Banach Center Publ.*, pages 391–400. PWN, Warsaw, 1989.

[39] W. Shevchik. Spectral Properties of Some Semi-Elliptic Operators in L_p Spaces. *Math. Nachr. 202*, 151-162, 1999.

[40] M. Solomyak. and E. Verbitsky On a spectral problem related to self-similar measures, *Bull. London Math. Soc. 27*, 242-248, 1995.

[41] P.M. Soardi. On nonisotropic Lipschitz spaces. In *Harmonic analysis (Cortona, 1982)*, volume 992 of *Lecture Notes in Math.*, pages 115–138. Springer, Berlin, 1983.

[42] E.M. Stein and S. Wainger. Problems in harmonic analysis related to curvature. *Bull. Amer. Math. Soc.*, 84(6):1239–1295, 1978.

[43] B. Stöckert and H. Triebel. Decomposition methods for function spaces of $B^s_{p,q}$ type and $F^s_{p,q}$ type. *Math. Nachr.*, 89:247–267, 1979.

[44] H. Triebel. *Fourier analysis and function spaces.* Teubner, Leipzig, 1977.

[45] H. Triebel. *Theory of Function Spaces.* Birkhäuser, Basel, 1983.

[46] H. Triebel. A priori estimates and boundary value problems for semi-elliptic differential equations: a model case. *Comm. in partial differential equations* , 8(15):1621–1644, 1983.

[47] H. Triebel. *Theory of Function Spaces II.* Birkhäuser, Basel, 1992.

[48] H. Triebel. *Higher analysis.* Barth Verlag, Leipzig, 1992.

[49] H. Triebel. *Fractals and Spectra.* Birkhäuser, Basel, 1997.

[50] H. Triebel. *The structure of functions.* Birkhäuser, Basel, 2001.

[51] H. Triebel. Towards a Gausslet analysis : Gaussian representations of functions. In M. Cwikel, M. Englis, A. Kufner, L.-E. Persson, and G. Sparr, editors, *Function Spaces, Interpolation Theory and Related Topics*, pages 425–450. Proceedings of the International Conference in honour of Jaak Peetre on his 65th birthday, Lund, Sweden August 17-22, 2000, de Gruyter Proceedings, 2002.

[52] H. Triebel. Wavelet frames for distributions; local and pointwise regularity. *Studia Math.*, 154(1):59–88, 2003.

[53] H. Triebel. Wavelet bases in anisotropic function spaces. Proc. *Function Spaces, Differential Operators and Nonlinear Analysis*, FSDONA-04, Milovy, Czech Republic 2004. Math. Inst. Acad. Sci. Czech Republic, Praha, 2005, 370-387

[54] H. Triebel. Approximation numbers in function spaces and the distribution of eigenvalues of some fractal elliptic operators. *J. Approx. Theory*, 129:1-27, 2004.

[55] H. Triebel. *Theory of Function Spaces III* , Birkhäuser, Basel, to appear 2006.

[56] P. Wojtaszczyk. *A mathematical introduction to wavelets*, volume 37 of *London Math. Soc. Student Texts.* Cambridge Univ. Press, Cambridge, 1997.

[57] M. Yamazaki. A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(1):131–174, 1986.

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